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Decentralised minimum-time consensus☆

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ABSTRACT

We consider the discrete-time dynamics of a network of agents that exchange information according to a nearest-neighbour protocol under which all agents are guaranteed to reach consensus asymptotically. We present a fully decentralised algorithm that allows any agent to compute the final consensus value of the whole network in finite time using the minimum number of successive values of its own state history. We show that the minimum number of steps is related to a Jordan block decomposition of the network dynamics, and present an algorithm to compute the final consensus value in the minimum number of steps by checking a rank condition of a Hankel matrix of local observations. Furthermore, we prove that the minimum number of steps is related to graph theoretical notions that can be directly computed from the Laplacian matrix of the graph and from the minimum external equitable partition.

1. Introduction

Fuelled by applications in a variety of fields, there has been a recent surge of interest in consensus dynamics (Cao, Ren, & Chen, 2012). In its most basic formulation, the consensus problem studies the linear discrete-time dynamics of a network of agents that exchange information according to the nearest-neighbour averaging rule. The consensus problem has broad implications beyond the analysis and design of collective behaviour in multi-agent systems. Various applications can be cast in this framework, including swarming and flocking (Tanner, Jadbabaie, & Pappas, 2003), distributed computing (Bertsekas & Tsitsiklis, 1989), agreement in social networks (Olfati-Saber, 2005; Watts & Strogatz, 1998) or synchronisation of coupled oscillators (Barahona & Pecora, 2002; Jadbabaie, Motee, & Barahona, 2004; Pecora & Barahona, 2005; Stan & Sepulchre, 2007).

The design of efficient distributed consensus algorithms is a current focus of active research in the Control literature. Under broad assumptions, well-known results (Jadbabaie, Lin, & Morse, 2003; Olfati-Saber & Murray, 2004; Ren & Beard, 2007) give conditions to ensure that the state of each agent reaches the consensus value *asymptotically*. From a practical point of view, however, requiring an 'infinite' or arbitrarily long time to obtain the final consensus value of the system is unsatisfactory. The principles for the computation of the asymptotic final value of the network in finite time were introduced in Sundaram and Hadjicostis (2007). Other work related to finite time consensus in continuous-time systems can be found in Wang and Xiao (2010) and Hui, Haddad, and Bhat (2008).

In Yuan (2012), Yuan, Liu, Murray, and Goncalves (2012) and Yuan, Stan, Shi, and Goncalves (2009), we extended the results in Sundaram and Hadjicostis (2007) and studied the *minimum* number of discrete-time steps required by an arbitrarily chosen agent to compute the asymptotic final value of the network without any prior knowledge of the system dynamics. Importantly, the information used for that purpose was solely based on the accumulation of the successive state values of the agent under consideration and, consequently, the corresponding algorithm was truly *decentralised*. Of related interest is Ref. (Bauso, Giarre, & Pesenti, 2009), which considers a game-theoretical approach to

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consensus when agents have prediction capabilities. The approach in Bauso et al. (2009) relies on the use of characteristic polynomials to obtain an upper bound on the number of steps required by any node to compute the consensus value.

The structure of this paper is as follows. Firstly, we introduce an algorithm that allows any agent in a consensus-guaranteed network to compute the consensus value using one step fewer than in Yuan et al. (2009) (provided that every node assumes consensus will be reached). This algorithm relies on the analysis of the rank of a Hankel matrix constructed from local observations at any chosen node. Furthermore, we show that the minimum number of steps is linked to a global property of the network: the degree of a specific matrix polynomial. This provides us with an algebraic characterisation of the local convergence to consensus in terms of properties of the Laplacian matrix of the graph. Finally, we show that the minimum number of steps required to compute the consensus value from local observations of any chosen node can be bounded in terms of two combinatorial graph theoretical properties: the minimum external equitable partition of the graph with respect to that node and the longest distance for that node. Throughout the paper we illustrate our results with examples that highlight how our framework can establish a link between the spectral and graph theoretical properties of a network of interacting agents and the minimum-time solution of distributed decision-making problems.

Notation. The notation in this paper is standard. For a matrix $A \in \mathbb{R}^{M\times N}$, $A[i,j] \in \mathbb{R}$ denotes the element in the ith row and jth column, $A[i,:] \in \mathbb{R}^{1\times N}$ denotes its ith row, $A[:,j] \in \mathbb{R}^{M\times 1}$ denotes its jth column, and $A[i_1 : i_2, j_1 : j_2]$ denotes the submatrix of A defined by the rows i_1 to i_2 and the columns j_1 to j_2. For a column vector $\alpha \in \mathbb{R}^{N\times 1}$, $\alpha[i]$ denotes its ith element. We denote by $e_r^T = [0,\ldots,0,1_{\text{rth}},0,\ldots,0] \in \mathbb{R}^{1\times N}$. Furthermore, I_N denotes the identity matrix of dimension N.

2. Consensus dynamics: formulation and previous results

2.1. Formulation of the problem

Consider a directed unweighted graph denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1,\ldots,v_n\}$ is the set of n nodes and $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$ is the set of edges. $W_{n\times n}$ is the corresponding adjacency matrix, with $W[i,j] = 1$ when there is a directed edge from j to i, and $W[i,j] = 0$ when there is no edge from j to i.

Let $x[i] \in \mathbb{R}$ denote the state of node i. The classical consensus problem for a network of continuous-time integrator individuals is defined by the following dynamics (Olfati-Saber & Murray, 2004):

$$\dot{x}(t) = -Lx(t),$$

where $L \in \mathbb{R}^{n\times n}$ is the Laplacian matrix induced by the topology $\mathcal{G}$. L is defined as $L[i,j] = -W[i,j] \forall i \neq j$, and $L[i,i] = \sum_{l\neq i} W[i,l]$.

In this paper we consider the associated discrete-time consensus dynamics on a network:

$$\begin{aligned} x_{k+1} &= (I_n - \epsilon L)\, x_k \triangleq A\, x_k \\ y_k &= e_r^T x_k = x_k[r], \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^{n\times 1}$ and ϵ is the sampling time. Without loss of generality, we concentrate on the case where the measurable output $y_k \in \mathbb{R}$ corresponds to the local state of an arbitrarily chosen individual node labelled r.

2.2. Global asymptotic convergence to distributed consensus (Jadbabaie et al., 2003); (Olfati-Saber & Murray, 2004)

Let $d_{\max} = \max_i L[i,i]$ denote the maximal node in-degree of the graph $\mathcal{G}$. If the network has a rooted directed spanning tree over time (Jadbabaie et al., 2003; Ren & Beard, 2007) (or if it is connected in the case of an undirected graph) and the sampling time ϵ is such that $0 < \epsilon < 1/d_{\max}$, then the discrete-time version of the classical consensus protocol (1) ensures global asymptotic convergence to consensus in the sense that

$$\lim_{k\to\infty} x_k = \left(c^T x_0\right)\, \mathbf{1}_{n\times 1}$$

where $\mathbf{1}_{n\times 1}$ is a vector with all components equal to 1, and c^T is a constant row vector. In other words, the states of all nodes converge asymptotically to the same value given by a linear combination of the initial states of the nodes, x_0.

Algebraic characterisation of distributed asymptotic consensus (Xiao & Boyd, 2004)

When $c^T\mathbf{1} = 1$, the iteration given by (1) achieves distributed consensus if and only if:

A.1. A has a simple eigenvalue at 1 and all other eigenvalues have a magnitude strictly less than 1.
A.2. The left and right eigenvectors of A corresponding to the eigenvalue 1 are c^T and $\mathbf{1}$, respectively.

2.3. Finite-time computation of the final consensus value (Sundaram & Hadjicostis, 2007)

Recent work by Sundaram and Hadjicostis (Sundaram & Hadjicostis, 2007) has shown that it is possible to obtain the final value of the consensus dynamics in a finite number of steps. Their result hinges on the use of the minimal polynomial associated with the consensus dynamics (1) in conjunction with the final value theorem.

Definition 1 (*Minimal Polynomial of a Matrix*)**.** The minimal polynomial of matrix $A \in \mathbb{R}^{n\times n}$ is the monic polynomial $q(t) \triangleq t^{D+1} + \sum_{i=0}^{D} \alpha_i t^i$ with minimum degree $D+1$ that satisfies $q(A) = 0$.

Given the explicit solution of (1) with initial state x_0, it follows from the definition of the minimal polynomial that the dynamics (1) satisfies the linear regression equation:

$$x_{k+D+1} + \alpha_D x_{k+D} + \cdots + \alpha_1 x_{k+1} + \alpha_0 x_k = 0, \quad \forall k \in \mathbb{N}. \tag{2}$$

Similarly, the regression equation for $y_k = x_k[r]$, the measurable output at node r, is determined by the minimal polynomial of the corresponding matrix observability pair $[A, e_r^T]$.

Definition 2 (*Minimal Polynomial of a Matrix Pair*)**.** The minimal polynomial associated with the matrix pair $[A, e_r^T]$ denoted by $q_r(t) \triangleq t^{D_r+1} + \sum_{i=0}^{D_r} \alpha_i^{(r)} t^i$ is the monic polynomial of minimum degree $D_r + 1 \leq D + 1$ that satisfies $e_r^T q_r(A) = 0$.

Remark 1. The minimal polynomial of a matrix and the minimal polynomial of a matrix pair are unique due to the monic property.

Again, it is straightforward to show that:

$$\sum_{i=0}^{D_r+1} \alpha_i^{(r)} y_{k+i} = 0, \quad \forall k \in \mathbb{N}, \tag{3}$$

where $\alpha_{D_r+1}^{(r)} = 1$. Hence each node r is associated with a particular length (D_r+1) of the regression in (3) which is upper bounded by $(D+1)$, the degree of the minimal polynomial of the dynamical matrix A.

Consider now the z-transform $Y(z) \triangleq \mathcal{Z}(y_k)$. From (3) and the time-shift property of the z-transform, it follows that:

$$Y(z) = \frac{\sum_{i=1}^{D_r+1} \alpha_i^{(r)} \sum_{\ell=0}^{i-1} y_\ell z^{i-\ell}}{q_r(z)} \triangleq \frac{H(z)}{q_r(z)}. \tag{4}$$

Under the assumptions specified in Section 2.2, the minimal polynomial $q_r(t)$ does not possess any unstable root apart from one at 1. We can then define the following polynomial:

$$p_r(z) \triangleq \frac{q_r(z)}{z-1} \triangleq \sum_{i=0}^{D_r} \beta_i z^i. \tag{5}$$

The application of the final value theorem (Gluskin, 2003) and some simple algebra then gives the consensus value as:

$$\phi = \lim_{z\to 1}(z-1)Y(z) = \frac{H(1)}{p_r(1)} = \frac{y_{D_r}^T \beta}{\mathbf{1}^T \beta} \tag{6}$$

where $y_{D_r}^T = \begin{bmatrix} y_0 & y_1 & \cdots & y_{D_r} \end{bmatrix}$ and $\beta_{(D_r+1)\times 1}$ is the vector of coefficients of the polynomial $p_r(z)$ defined in (5).

Based on these results, an algorithm to obtain the consensus value was proposed in Sundaram and Hadjicostis (2007). The algorithm in Sundaram and Hadjicostis (2007) is distributed but not entirely local, in the sense that a local calculation is repeated over n independent iterations (where n is the number of nodes of the network) and at each iteration each node needs to store its own values for the past $n+1$ steps. Hence a total of $n(n+1)$ successive values of $x[r]$ are required for the calculation of the consensus value ϕ using the procedure described in Sundaram and Hadjicostis (2007).

2.4. Minimum-time, decentralised computation of the final consensus value

The purpose of this paper is to characterise the computation of the final consensus value ϕ using only the output observations of node r *in minimum time*. We formalise and improve our previous results (Yuan et al., 2009) and show that, for a general arbitrary initial condition (i.e., except for a set of initial conditions with Lebesgue measure zero (Blondel, Hendrickx, & Tsitsiklis, 2010)), the consensus value can be obtained from local observations in a minimum number of steps that does not depend explicitly on the total size of the graph. In our framework, the minimum number of steps is computed in a truly decentralised manner by checking a rank condition for a Hankel matrix constructed exclusively from local output observations. We also provide a graph theoretical characterisation of this local property in terms of properties of the graph Laplacian and its minimum external equitable partition. This characterisation can be used to provide further understanding into which graph properties contribute to the disparity in the ability of different nodes to compute the global consensus value from local information.

3. Minimum time consensus and the Jordan block decomposition of the consensus dynamics

Given the linear system (1) and an initial state x_0, it follows from above that there always exist scalars $d \triangleq d(r, x_0) \in \mathbb{N}$ and $a_0, \ldots, a_d \in \mathbb{R}$ such that the following linear regression equation is satisfied $\forall k \in \mathbb{N}$

$$y_{k+d+1} + a_d y_{k+d} + \cdots + a_1 y_{k+1} + a_0 y_k = 0. \tag{7}$$

From the definitions above, it is clear that $D_r + 1$ is the *minimum length of recursion*:

$$D_r + 1 = \min_{d\in\mathbb{N}} \max_{x_0\in\mathbb{R}^n} \{d(r, x_0) + 1 : \text{Eq. (7) holds} \forall k\}.$$

Remark 2. Among the many recursions of length d that are not necessarily minimum, $(D_r + 1)$ appears as a min–max over the space of (d, x_0). When $d + 1 = D_r + 1$, the coefficients a_i in (7) correspond to $\alpha_i^{(r)}$, the coefficients of the minimal polynomial of the matrix pair $[A, e_r^T]$ in (3).

In this section, we give an algebraic characterisation of the minimum number of steps $D_r + 1$ based on the projection of the Jordan block decomposition of A^k on e_r^T. Our aim is to obtain the coefficients $\alpha_i^{(r)}$ in (3) from stored data so that we can compute future outputs recursively.

Consider the standard Jordan decomposition of A:

$$A = SJS^{-1} \tag{8}$$

$$S = \begin{bmatrix} s_1 & s_2 & \cdots & s_n \end{bmatrix} \tag{9}$$

$$J = \operatorname{diag}\{J_1(\lambda_1), J_2(\lambda_2), \ldots, J_l(\lambda_l)\} \tag{10}$$

where

$$J_i(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix}_{n_i\times n_i}, \tag{11}$$

and s_i, the columns of the non singular matrix S, are the generalised eigenvectors of A (Zhou, Doyle, & Glover, 1996). The matrix A has l (not necessarily distinct) eigenvalues λ_i, each of them associated with a Jordan block of size n_i, such that $\sum_{i=1}^{l} n_i = n$. Without loss of generality, we assume that the blocks are ordered by decreasing size: $n_1 \geq n_2 \geq \cdots \geq n_l$.

Using Eq. (8), the linear dynamics (1) can be rewritten as follows:

$$x_k[r] = e_r^T A^k x_0 = \left(e_r^T S\right) J^k \left(S^{-1} x_0\right) \triangleq \sigma^T J^k \chi, \tag{12}$$

where the vectors

$$\sigma^T = \begin{bmatrix} \sigma_1^T & \sigma_2^T & \cdots & \sigma_l^T \end{bmatrix}_{1\times n} \tag{13}$$

$$\chi^T = \begin{bmatrix} \chi_1^T & \chi_2^T & \cdots & \chi_l^T \end{bmatrix}_{1\times n} \tag{14}$$

are partitioned according to the Jordan blocks in (8), e.g., $\sigma_1^T = \begin{bmatrix} \sigma_{11} \cdots \sigma_{1n_i} \end{bmatrix}$ and $\chi_1^T = \begin{bmatrix} \chi_{11} \cdots \chi_{1n_1} \end{bmatrix}$. Here,

$$J^k = \operatorname{diag}\left\{J_1^k(\lambda_1), J_2^k(\lambda_2), \ldots, J_l^k(\lambda_l)\right\}$$

has the well-known structure (Horn & Johnson, 1999):

$$J_i^k(\lambda_i) = \sum_{m=0}^{k-1} \binom{k}{m} \lambda_i^{k-m} J_i^m(0), \tag{15}$$

where $J_i^m(0)$ follows from the definition of the Jordan block in (11).

The output dynamics (12) then becomes:

$$x_k[r] = \sum_{i=1}^{l} \sum_{m=0}^{k-1} \binom{k}{m} \lambda_i^{k-m} \left[\sigma_i^T J_i^m(0)\, \chi_i\right]. \tag{16}$$

Note that because of its Jordan block structure, the matrix $J_i^m(0)$ induces a strict m-shift on the vector χ_i for $m \leq n_i$. Therefore if $k \geq \max_i\{n_i\}$, we have:

$$\begin{aligned} x_k[r] &= \sum_{i=1}^{l} \sum_{m=0}^{n_i-1} \binom{k}{m} \lambda_i^{k-m} \left[\sum_{j=1}^{n_i-m} \sigma_{ij} \chi_{ij+m}\right] \\ &\triangleq \sum_{i=1}^{l} \sum_{m=0}^{n_i-1} \binom{k}{m} \lambda_i^{k-m} g_{im}. \end{aligned} \tag{17}$$

However, some of the g_{im} might be zero (we might even have situations in which all the coefficients associated with a particular eigenvalue are zero) so that the dynamics of node r can be written as:

$$x_k[r] = \sum_{i=1}^{l_r} \sum_{m=0}^{n_i^r-1} \binom{k}{m} \lambda_i^{k-m} g_{im} \tag{18}$$

where $n_i^r \leq n_i$ and $l_r \leq l$. Eq. (18) can be rewritten as a dot product:

$$x_k[r] = \boldsymbol{v_r}(k)^T \boldsymbol{g}_r \triangleq \begin{bmatrix} v_1^T(k) & v_2^T(k) & \cdots & v_{l_r}^T(k) \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{l_r} \end{bmatrix}$$

where

$$v_i^T(k) \triangleq \begin{bmatrix} \binom{k}{0}\lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \cdots & \binom{k}{n_i^r-1}\lambda_i^{k-n_i^r+1} \end{bmatrix}_{1\times n_i^r}$$

$$g_i^T \triangleq \begin{bmatrix} g_{i0} & \cdots & g_{i(n_i^r-1)} \end{bmatrix}.$$

Here, $\{\lambda_1, \ldots, \lambda_{l_r}\}$ is an ordered subset of *distinct* eigenvalues from the original Jordan block decomposition and g_i are the coefficients in Eq. (17). The degree of the characteristic polynomial that underlies the length of the recursion for node r is then:

$$\sum_{i=1}^{l_r} n_i^r = D_r + 1.$$

Remark 3. Note from Eq. (12) that n_i^r depends on e_r, A, and x_0.

Based upon the decomposition of confluent Vandermonde matrices introduced in Boley, Luk, and Vandevoorde (1998), it is easy to see that

$$v_i^T(k) = \bar{e}_i^T \mathcal{J}_i^k(\lambda_i)$$

where $\mathcal{J}_i(\lambda_i)$ is a Jordan block of size n_i^r as defined in (11) and $\bar{e}_i^T = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}_{1\times n_i^r}$ is the unit vector of the same length. The dynamics (12) can thus be rewritten in terms of a Jordan decomposition of reduced dimensionality as follows:

$$x_k[r] = \boldsymbol{E}_r^T \boldsymbol{J}_r^k \boldsymbol{g}_r, \quad \forall k, \tag{19}$$

where

$$\boldsymbol{E}_r^T \triangleq \begin{bmatrix} \bar{e}_1^T & \cdots & \bar{e}_{l_r}^T \end{bmatrix}_{1\times(D_r+1)} \quad \text{and}$$

$$\boldsymbol{J}_r \triangleq \operatorname{diag}\left\{\mathcal{J}_1(\lambda_1), \mathcal{J}_2(\lambda_2), \ldots, \mathcal{J}_{l_r}(\lambda_{l_r})\right\} \tag{20}$$

are partitioned according to the l_r blocks.

From the above analysis we have the following lemma.

Lemma 1. *Consider the discrete-time LTI system (1). The minimal polynomial associated with $x[r]$, as given in Definition 2, is the characteristic polynomial of the matrix $\boldsymbol{J}_r$ in Eq. (19) which has order $D_r + 1 = \sum_{i=1}^{l_r} n_i^r$. The final consensus value ϕ can be computed from Eq. (6) based on the coefficients of the minimal polynomial of the pair $[A, e_r^T]$ and the successive values of $x[r]$.*

Proof. The Jordan matrix $\boldsymbol{J}_r$ in Eq. (19) has the property that each of its Jordan blocks has distinct eigenvalues. It then follows (see Horn and Johnson (1999)) that the minimal polynomial of $[A, e_r^T]$ is the same as the characteristic polynomial of $[\boldsymbol{J}_r, e_r^T]$, i.e., $e_r^T q_r(A) = e_r^T q_r(\boldsymbol{J}_r)$. Therefore the minimal polynomial possesses the following explicit form: $\det(\boldsymbol{J}_r - tI) = \prod_{i=1}^{l_r}(t-\lambda_i)^{n_i^r} = t^{D_r+1} + \alpha_{D_r} t^{D_r} + \cdots + \alpha_1 t + \alpha_0$, and has degree $D_r + 1$. This latter relationship also shows that $D_r + 1 = \sum_{i=1}^{l_r} n_i^r$. □

Remark 4. Lemma 1 states that, instead of being written in terms of an n-dimensional Jordan block decomposition of J, as in Eq. (12), the general expression of $x_k[r]$ can be equivalently written in terms of a smaller $D_r + 1$-dimensional Jordan matrix $\boldsymbol{J}_r$, as in Eq. (19).

Remark 5. The minimum integer value $D_r + 1$ necessary for the recursion (7) to hold for a generic initial condition x_0 is given by the degree of the minimal polynomial of the observability pair $[A, e_r^T]$ (see Yuan et al. (2009)). In other words, Eq. (7) holds for a randomly chosen initial state x_0, except for a set of initial conditions of Lebesgue measure zero (Blondel et al., 2010).

4. Decentralised minimum-time consensus computation algorithm

In the decentralised problem we assume that node r does not have access to any external information, such as the total number of individuals n in the network, the number of its neighbours, its local communication links with its neighbours, or the state values of its neighbours. In Yuan et al. (2009), we showed that for a general discrete-time LTI system (1), $2D_r + 3$ successive discrete-time steps are needed by an individual r to compute the final value in a fully decentralised manner. If the network is well-designed for consensus (i.e., if the assumptions in Section 2.2 are satisfied and asymptotic convergence to consensus is guaranteed), we hereby propose an algorithm (Algorithm 1) that computes the final value using $2D_r + 2$ successive discrete-time steps.

Problem 1 (*Decentralised Problem*)**.** Consider the discrete-time LTI dynamics in Eq. (1) where an arbitrarily chosen state $x[r]$ is observed and assume that the conditions for consensus (Assumptions A.1 and A.2) are satisfied. The decentralised problem is to compute in finite time the asymptotic value of this state $\phi = \lim_{k\to\infty} x_k[r]$ using only its own successive state values $y_k = x_k[r]$, observed over a range of time-steps which is minimum.

Consider the vector of $2k+1$ successive discrete-time values at node r, $X_{0,1,\ldots,2k}[r] = (x_0[r], x_1[r], \ldots, x_{2k}[r])$, and its associated Hankel matrix:

$$\Gamma\{X_{0,1,\ldots,2k}[r]\} \triangleq \begin{bmatrix} x_0[r] & x_1[r] & \cdots & x_k[r] \\ x_1[r] & x_2[r] & \cdots & x_{k+1}[r] \\ \vdots & \vdots & \ddots & \vdots \\ x_k[r] & x_{k+1}[r] & \cdots & x_{2k}[r] \end{bmatrix}$$

$$k \in \mathbb{N}. \tag{21}$$

Consider also the vector of differences between successive values of $x[r]$:

$$\overline{X}_{0,1,\ldots,2k}[r] = \{x_1[r] - x_0[r], \ldots, x_{2k+1}[r] - x_{2k}[r]\}.$$

Notice that under the assumption that the network will reach consensus, the final value of all nodes will eventually be the same and, as result, the computed final value is the consensus value. Under this assumption, the z-transform of Eq. (3) has a root at 1, as discussed in Eq. (5). We use this fact in Algorithm 1 below to reduce the number of steps required to compute the final consensus value to $2D_r + 2$ steps.

Algorithm 1 Decentralised minimum-time consensus value computation

Data: Successive observations of $x_i[r]$, $i = 0, 1, \ldots$.
Result: Final consensus value: ϕ.
Step 1 Compute the vector of differences $\overline{X}_{0,1,\ldots,2k}$. Increase the dimension k of the square Hankel matrix $\Gamma\{\overline{X}_{0,1,\ldots,2k}[r]\}$ until it loses rank and store the first defective Hankel matrix.
Step 2 The normalised kernel $\beta = \begin{bmatrix} \beta_0 & \ldots & \beta_{D_r-1} & 1 \end{bmatrix}^T$ of the first defective Hankel matrix gives the coefficients of Eq. (6).
Step 3 Compute the final consensus value ϕ using Eq. (6).

To understand Algorithm 1, consider a Vandermonde factorisation (Boley et al., 1998) of the Hankel matrix (21):

$$\Gamma\{X_{0,1,\ldots,2k}[r]\} = V(0,k) T_r V^T(0,k), \tag{22}$$

in which we have defined the confluent Vandermonde matrix

$$V(0,k)_{(k+1)\times(D_r+1)} = \begin{bmatrix} \boldsymbol{E}_r^T \\ \boldsymbol{E}_r^T \boldsymbol{J}_r \\ \vdots \\ \boldsymbol{E}_r^T \boldsymbol{J}_r^k \end{bmatrix}, \tag{23}$$

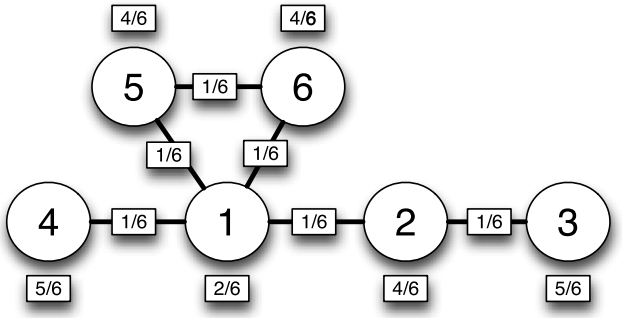

Fig. 1. Underlying topology for Example 1 with sampling time $\epsilon = 1/6$.

in terms of the elements defined in Eq. (20). As shown in Boley et al. (1998), the $(D_r + 1) \times (D_r + 1)$ block diagonal matrix

$$T_r = \text{diag}\{T_{r,1}, \ldots, T_{r,l_r}\}, \quad T_{r,i} \in \mathbb{R}^{n_i^r \times n_i^r},$$

has the following symmetric upper anti-diagonal form:

$$T_{r,i} = \begin{bmatrix} * & * & * & * & t_i \\ * & * & * & t_i & \\ * & * & \iddots & & \\ * & t_i & & 0 & \\ t_i & & & & \end{bmatrix},$$

where t_i and $*$ are determined from the values of $x_k[r]$.

Without loss of generality, consider $\lambda_1 = 1$ so that $T_{r,1} \in \mathbb{R}$. We then have

$$\begin{aligned} \Gamma\{\overline{X}_{0,1,\ldots,2k}[r]\} &= \Gamma\{X_{1,2,\ldots,2k+1}[r]\} - \Gamma\{X_{0,1,\ldots,2k}[r]\} \\ &= VT_r\text{diag}\{\lambda_1, \ldots, \lambda_{l_r}\}V^T - VT_rV^T \\ &= VT_r\text{diag}\{0, \lambda_2 - 1, \ldots, \lambda_{l_r} - 1\}V^T \\ &= V\text{diag}\{0, (\lambda_2 - 1)T_{r,2}, \ldots, (\lambda_{l_r} - 1)T_{r,l_r}\}V^T \\ &= V'\text{diag}\{(\lambda_2 - 1)T_{r,2}, \ldots, (\lambda_{l_r} - 1)T_{r,l_r}\}V'^T, \end{aligned}$$

where $V' = V[2 : k + 1, 2 : D_r + 1]$. From the last equation, it is easy to see that $k \geq D_r + 1$ is a necessary and sufficient condition for $\Gamma\{\overline{X}_{0,1,\ldots,2k}[r]\}$ to be defective.

Theorem 1. *Consider the system in (1) and assume that the conditions for consensus (Assumptions* A.1 *and* A.2*) are satisfied. Then the minimum number of successive discrete-time steps, starting from step i, for the arbitrarily chosen node $x[r]$ to compute its final consensus value is $2(D_r+1)-\delta_r-\min\{i, \delta_r\}$, where δ_r is the number of zero roots in $q_r(t) = 0$.*

Proof. The proof follows from the above derivations and Corollary 1 of Ref. (Yuan et al., 2009) by taking $z_k \triangleq x_{k+1}[r] - x_k[r]$ as y_k in that Corollary. □

For simplicity of exposition, we make the following assumption in the rest of this section:

A.3. The matrix A in (1) does not possess any eigenvalue at 0.

Remark 6. Under Assumption A.3, Theorem 1 establishes that the minimum number of steps for node r to compute the final consensus value is $2D_r + 2$.

Example 1. Consider the network topology in Fig. 1 under dynamics (1) with $A \triangleq I_n - \epsilon L$ and a sampling time $\epsilon = 1/6$. The topology is undirected and connected and A satisfies Assumptions A.1–A.3. Therefore the final value of each node is the average of the initial state values. For the randomly chosen initial state $x_0 = \begin{bmatrix}1.3389 & 2.0227 & 1.9872 & 6.0379 & 2.7219 & 1.9881\end{bmatrix}^T$, the final consensus value is thus 2.6828. We now apply Algorithm 1 to node $r = 1$.

Table 1
Comparison of the minimum number of successive values needed by each node to compute the final consensus value of the network in Fig. 1 with $n = 6$ nodes.

	Ref. (Sundaram & Hadjicostis, 2007)	Our result
Node 1	$6 \times 7 = 42$	$2 \times 4 = 8$
Node 2	$6 \times 7 = 42$	$2 \times 4 = 8$
Node 3	$6 \times 7 = 42$	$2 \times 4 = 8$
Node 4	$6 \times 7 = 42$	$2 \times 5 = 10$
Node 5	$6 \times 7 = 42$	$2 \times 6 = 12$
Node 6	$6 \times 7 = 42$	$2 \times 6 = 12$

step 1. *We increase the dimension k of the square Hankel matrix $\Gamma\{\overline{X}_{0,1,\ldots,2k}[1]\}$ until it loses rank. This happens for $k = 3$. We then store the first defective Hankel matrix*:

$$\Gamma\{\overline{X}_{0,1,\ldots,8}[1]\} = \begin{bmatrix} 1.2358 & 0.2050 & 0.0367 & 0.0047 \\ 0.2050 & 0.0367 & 0.0047 & -0.0037 \\ 0.0367 & 0.0047 & -0.0037 & -0.0067 \\ 0.0047 & -0.0037 & -0.0067 & -0.0079 \end{bmatrix}.$$

step 2. *The normalised kernel of the first defective Hankel matrix is*

$$\beta = \begin{bmatrix} -0.0833 & 0.7778 & -1.6667 & 1 \end{bmatrix}^T.$$

This gives the coefficients of Eq. (6).

step 3. *We compute the final consensus value $\phi = 2.6828$ using* Eq. (6).

The value of ϕ obtained in a decentralised manner is equal to the consensus value given by the average of the initial state. Repeating the procedure for each of the six nodes gives the same value ϕ. However, the number of steps required by each node to compute the final consensus value ϕ differs. This is summarised in Table 1.

While the method proposed in Sundaram and Hadjicostis (2007) requires a total of $n(n + 1)$ successive values of $x[r]$, our algorithm shows that the minimum number of successive values of $x[r]$ needed is *node specific* and is just $2(D_r + 1)$ for *almost all* initial conditions. Furthermore, our algorithm is completely decentralised, i.e., it does not require any knowledge of the total number of nodes in the network, n, or of any other global (centralised) information about the network (contrary to Sundaram and Hadjicostis (2007, Section V)).

As can be noted in Table 1, some nodes need fewer successive observations of their own state to compute the final consensus value of the network. We call such nodes *dominant nodes*. A question arises at this point: given a consensus-guaranteed network, can we identify the dominant nodes? We address this question in the following section using graph-theoretical concepts.

5. Graph-theoretical characterisation of the minimum number of steps

We now provide answers to the question raised at the end of the last section. We do this from two different perspectives. In Section 5.1, we provide an algebraic characterisation of the minimum recursion length $D_r + 1$ based on the (grounded) graph Laplacian. In Section 5.2, we link $D_r + 1$ to the number of cells in a special partition of the graph called the *minimum external equitable partition with respect to node r*.

For simplicity of exposition, we only consider undirected graphs in the following sections, i.e., hereafter we assume:

A.4. The matrices L and A in Eq. (1) are symmetric.

5.1. Algebraic graph-theoretical characterisation

We start by stating the connection between the minimum recursion length at node r and the rank of the observability matrix.

Proposition 1. *Consider the observability matrices*

$$\Omega_r = \begin{bmatrix} e_r^T \\ e_r^T A \\ \cdots \\ e_r^T A^{n-1} \end{bmatrix} \quad \text{and} \quad \Theta_r = \begin{bmatrix} e_r^T \\ e_r^T L \\ \cdots \\ e_r^T L^{n-1} \end{bmatrix}. \tag{24}$$

Then $D_r + 1 = \operatorname{rank}(\Omega_r) = \operatorname{rank}(\Theta_r)$.

Proof. The first equality follows directly from the definition of the minimal polynomial (Definition 2) and the Cayley–Hamilton theorem (Zhou et al., 1996):

$$e_r^T q_r(A) = \sum_{i=0}^{D_r+1} \alpha_i^{(r)} e_r^T A^i = 0.$$

Hence the number of independent rows of the matrix Ω_r is precisely $D_r + 1$. The second equality follows from the definition $A = I - \epsilon L$ and Gaussian elimination (Horn & Johnson, 1999). □

These results imply that $D_r + 1$ is related to the number of (distinct) eigenvalues of the Laplacian matrix whose eigenvectors have non-zero components for node r.

Proposition 2. *Consider the system in Eq. (1) satisfying Assumptions* A.1–A.4. *Additionally, assume that the Laplacian matrix L has no repeated eigenvalues. Then*

$$D_r + 1 = n - \eta_r,$$

where η_r is the number of eigenvectors of L with a 0 at the rth component.

Proof. Without loss of generality, assume we observe node 1, i.e., let $r = 1$ and η_1 be the number of eigenvectors of L with a zero in their first component:

$$Lv_i = \lambda_i v_i, \qquad v_i^T = [0, u_i^T], \quad i = 1, \ldots, \eta_1.$$

It then follows that

$$e_1^T L^k v_i = 0, \quad \forall k, \quad \text{and} \quad \Theta_1 v_i = 0 \quad \text{for } i = 1, \ldots, \eta_1.$$

Since $L = L^T$ and the eigenvectors form an orthogonal basis, then $\dim(\ker(\Theta_1)) \geq \eta_1$. Conversely, it is easy to see that a necessary condition for a vector to belong to the kernel of Θ_1 is that its first component is zero: $\Theta_1 v = 0 \Rightarrow e_1^T v = v[1] = 0$. Since the eigenvectors are all orthogonal and non-degenerate, this implies that $\dim(\ker(\Theta_1)) = \eta_1$ and therefore $\operatorname{rank}(\Theta_1) = n - \eta_1$. This result also follows from the PBH test (Zhou et al., 1996). □

Remark 7. If there are repeated eigenvalues, the above result just provides an upper bound:

$$D_r + 1 = \operatorname{rank}(\Theta_1) \leq n - \eta_1,$$

since it is possible to generate independent vectors with a zero rth component through linear combinations of the eigenvectors within each degenerate subspace. Hence, the number of repeated eigenvalues has to be further discounted from the nullity of the observability matrix, as well as any block of degenerate eigenvectors with a zero overall component in the rth position. This statement is linked to results on the dimensionality of the observable subspace. Below we provide a characterisation in terms of the grounded Laplacian that discounts the effect of eigenvalue multiplicity.

Our further algebraic characterisation relates $D_r + 1$ to the number of eigenvalues shared by the Laplacian matrix and the r-grounded Laplacian matrix.

Definition 3 (*Grounded Laplacian Matrix*)**.** Let $L \in \mathbb{R}^{n\times n}$ be the Laplacian matrix of graph $\mathcal{G}$. The r-grounded Laplacian matrix, denoted L_r, is the symmetric submatrix of L obtained by deleting the rth row and the rth column.

Remark 8. It is easy to show that any Laplacian matrix L can be written in terms of L_1 as

$$L = \begin{bmatrix} 1^T L_1 1 & -1^T L_1 \\ -L_1 1 & L_1 \end{bmatrix}. \tag{25}$$

Theorem 2. *Consider the system in Eq. (1) satisfying Assumptions* A.1–A.4. *Then*

$$D_r + 1 = n - \mu_r,$$

where μ_r is the number of eigenvalues shared between L and L_r.

Proof. Again, without loss of generality, let $r = 1$. Let $\lambda_i(L)$ be an eigenvalue of L with eigenvector v_i:

$$\begin{bmatrix} 1^T L_1 1 & -1^T L_1 \\ -L_1 1 & L_1 \end{bmatrix} \begin{bmatrix} v_i[1] \\ v_i[2:n] \end{bmatrix} = \lambda_i \begin{bmatrix} v_i[1] \\ v_i[2:n] \end{bmatrix}. \tag{26}$$

If $v_i[1] = 0$, then $u_i = v_i[2:n]$ is an eigenvector of L_1:

$$L_1 u_i = \lambda_i u_i \quad \text{and} \quad 1^T u_i = 0.$$

Conversely, consider a shared eigenvalue of L_1 and L: $\lambda_i(L_1) = \lambda_i(L) \neq 0$ with eigenvectors $u_i \in \mathbb{R}^{(n-1)\times 1}$ and $v_i \in \mathbb{R}^{n\times 1}$, respectively:

$$(1^T L_1 1)\, v_i[1] - 1^T L_1 v_i[2:n] = \lambda_i\, v_i[1] \tag{27}$$

$$-v_i[1]\, L_1 1 + L_1 v_i[2:n] = \lambda_i\, v_i[2:n] \tag{28}$$

$$L_1 u_i = \lambda_i u_i. \tag{29}$$

Due to the symmetry of L, multiplying Eq. (28) from the left with u_i^T, leads to:

$$v_i[1]\, u_i^T L_1 1 = 0 \Rightarrow \begin{cases} \text{(i) } v_i[1] = 0 \\ \text{(ii) } u_i^T L_1 1 = 0. \end{cases}$$

In both cases, there is an eigenvector of L with eigenvalue λ_i and a zero first component: for (i), this eigenvector is v_i with $v_i[1] = 0$; for (ii), this eigenvector is $[0, u_i^T]^T$. Hence, an eigenvalue of L with a zero rth component will be shared with the grounded Laplacian and, conversely, a shared eigenvalue implies the existence of an eigenvector of L with a zero rth component (which could potentially correspond to a linear combination of degenerate eigenvectors of L that did not have this property).

It then follows from the proof of Proposition 2 that $\dim(\ker(\Theta_1)) = \mu_1$, i.e., the dimension of the observable subspace from node r is equal to the number of shared eigenvalues between the Laplacian and r-grounded Laplacian, which is itself equal to $D_r + 1$. □

As shown above, the coincidence of eigenvalues between L and L_r discounts the effect of the degeneracy in the spectrum of L, as can also be seen from the following bound obtained from the interlacing theorem (Horn & Johnson, 1999):

Lemma 2. *Consider the system in Eq. (1) satisfying Assumptions* A.1–A.4. *If L has ℓ distinct eigenvalues λ_i with multiplicities m_i ($i = 1, \ldots, \ell$), then*

$$D_r + 1 \leq n - \sum_{i=1}^{\ell} (m_i - 1) = \ell.$$

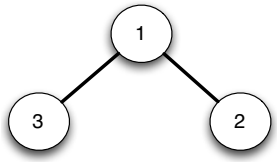

Fig. 2. Example graph used to illustrate the algebraic interpretation of the minimum number of steps.

Proof. From the interlacing theorem, if $\lambda_i(L)$ has multiplicity m_i, then λ_i is also an eigenvalue of L_r with multiplicity at least $(m_i - 1)$. Therefore, the number of shared eigenvalues between L and L_r is bounded by $\mu_r \geq \sum_{i=1}^{\ell}(m_i - 1) = n - \ell$. □

This result confirms our Remark 7 and shows that eigenvalue multiplicity needs to be discounted, i.e., the number of distinct eigenvalues of L provides an upper bound for $D_r + 1$.

Example 2. Consider the network in Fig. 2 with

$$L = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

The observability pair relative to node $r = 1$ based on Eq. (1) with $A = I - \frac{1}{3}L$ is $[A, [1\,0\,0]]$ with minimal polynomial $q_1(t) = t^2 - t$. Hence, $D_1 + 1 = 2$ since the order of this polynomial is 2.

The observability matrix in this case is

$$\Theta_1 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

From Proposition 1, it follows that $D_1 + 1 = \text{rank}(\Theta_1) = 2$, as expected.

From Theorem 2, note also that L and L_1 share the eigenvalue 1, hence $D_1 + 1 = 3 - 1 = 2$.

5.2. A non-algebraic, graph-theoretical characterisation

In this section, we consider the following question: given an undirected network, can we identify the dominant node(s) from the graph without any algebraic computation?

We adopt definitions and notations from Egerstedt (2010). A partition of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is defined as a mapping from vertices to subsets of vertices called *cells*: $\pi : \mathcal{V} \to \{C_1, \ldots, C_K\}$ where $C_i \subseteq \mathcal{V}$, $\forall i$. Let $Im(\pi) = \{C_1, \ldots, C_K\}$ denote the image of π, and $\deg_\pi(i, C_j)$ denote the node-to-cell degree, i.e., the number of nodes in cell C_j that share an edge with node v_i under partition π:

$$\deg_\pi(i, C_j) = \text{card}\left\{k \in \mathcal{V} | \pi(k) = C_j \text{ and } (i, k) \in \mathcal{E}\right\}.$$

We define $\pi^{-1}(C_i) = \{j \in \mathcal{V} | \pi(j) = C_i\}$, i.e., the set of nodes that are mapped to cell C_i.[2]

In what follows, we use the concept of external equitable partition (EEP) (Egerstedt, 2010). As we show below, EEPs correspond to partitions of the graph that disregard the internal interconnection structure inside a cell. We shall show that the EEP with respect to a node is directly related to the minimum number of steps necessary for this node to calculate the final consensus value.

[2] Note that π is not a one-to-one mapping but a many-to-one mapping. However, we can still define a new function to map back from C_j to $\mathcal{V}$ (Egerstedt, 2010).

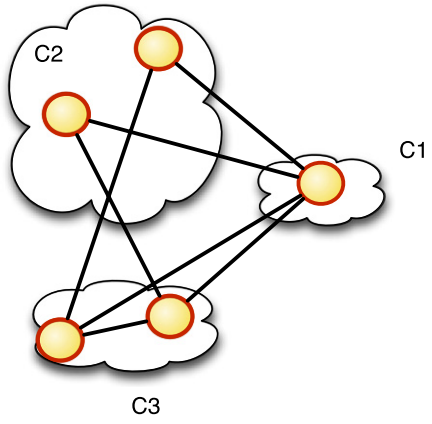

Fig. 3. An example to illustrate the external equitable partition (EEP). In this case, the partition is EEP with respect to node 1 in C_1.

Definition 4 (*External Equitable Partition (EEP) (Egerstedt, 2010)*)**.** A partition π^* of the set of nodes $\mathcal{V}$ consisting of $s > 1$ cells $\{C_1, \ldots, C_s\}$ is external equitable if the number of neighbours in C_j of a vertex $v \in C_i$ depends only on the choice of C_i and C_j $(i \neq j)$, i.e.,

$$\deg_{\pi^*}(l, C_j) = \deg_{\pi^*}(k, C_j), \quad \forall k, l \in \pi^{*-1}(C_i).$$

Definition 5 (*Minimum EEP with Respect to a Node*)**.** A partition π_r of $\mathcal{V}$ with cells $\{C_1, \ldots, C_s\}$ is external equitable with respect to node r if the partition is external equitable and the node r is in a cell alone, i.e., $\pi(v_r) = v_r$. The minimum EEP of a graph with respect to node r, π_r^*, is such that $\text{card}\{\text{Im}(\pi_r^*)\}$ is minimal.

Example 3. We illustrate the above definitions in Fig. 3. The partition shown is external equitable since different nodes in the same cell have the same degree to other cells, and is also external with respect to the node in C_1.

Theorem 3. *Consider the system in (1). Solely based on observations of node r, the minimum length of recursion necessary to obtain the final consensus value is less than the number of cells s_r in π_r^*, the minimum external equitable partition with respect to node r,*

$$d_r + 1 \leq D_r + 1 \leq card\left\{Im\left(\pi_r^*\right)\right\} \triangleq s_r, \tag{30}$$

where $d_r + 1$ is the longest distance from node r to any other node in the graph $\mathcal{G}$.

Proof. The proof that $d_r + 1 \leq D_r + 1$ is provided in Yuan (2009).

We now prove $D_r + 1 \leq s_r$. Without loss of generality, let $r = 1$. We use a Breadth-First-Search (BFS) algorithm to label the cells, as follows. We start from node 1 (i.e., cell 1) and explore all the neighbouring cells. For each of those nearest cells, we consider their own neighbouring cells and so on, until we have labelled all the cells in the minimum EEP with respect to node 1.

Consider now the block matrix obtained by permuting and partitioning A according to π_1^*:

$$A_{\pi_1^*} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s_1} \\ A_{21} & A_{22} & \cdots & A_{2s_1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s_1 1} & A_{s_1 2} & \cdots & A_{s_1 s_1} \end{bmatrix}.$$

Here, $A_{ii} \in \mathbb{R}^{l_i \times l_i}$ contains the interconnections between any two nodes in cell C_i^*, and l_i denotes the number of nodes in cell C_i^*.

Hence, $l_1 = 1$ and $\sum_{i=1}^{s_1} l_i = n$. The off-diagonal submatrices $A_{ij} \in \mathbb{R}^{l_i \times l_j}$ contain the interconnections between nodes in C_i^* and C_j^*. In particular, we shall consider the following submatrices:

$$A_1 \triangleq A_{\pi_1^*}[2:n, 2:n]$$
$$f_1^T \triangleq A_{\pi_1^*}[1, 2:n] = \begin{bmatrix} A_{12} & \cdots & A_{1j} & 0 & \cdots & 0 \end{bmatrix}.$$

Note that there are only j neighbouring cells to cell 1: $A_{1(j+1)}, \ldots, A_{1s_1} = 0$ for some $j > 1$.

The observability matrix (24) associated with the pair $[A_{\pi_1^*}, e_1^T]$ is:

$$\Omega_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ A_{11} & A_{12} & \cdots & A_{1s_1} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{bmatrix}, \tag{31}$$

where $*$ is a placeholder representing a real value. Applying Gaussian elimination to Ω_1, one can show that

$$\operatorname{rank}(\Omega_1) = \operatorname{rank}\left(\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \Xi & \\ 0 & & & \end{bmatrix}\right) = \operatorname{rank}(\Xi) + 1,$$

where Ξ is the observability matrix associated with the pair $[A_1, f_1^T]$. According to Camlibel, Zhang, and Cao (2012), Egerstedt (2010) and Martini, Egerstedt, and Bicchi (2010); Martini, Egerstedt, Cao, Camlibel, and Bicchi (in press), the rank of this observability matrix fulfills the following inequality:

$$\operatorname{rank}(\Xi) \leq \text{dim-span}\left\{\begin{bmatrix} \mathbf{1}_{r_2} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \mathbf{1}_{r_3} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \mathbf{1}_{r_{s_1}} \end{bmatrix}\right\}, \tag{32}$$

with $r_i = \operatorname{card}\{C_i^*\}$. Hence, $\operatorname{rank}(\Xi) \leq s_1 - 1$, from where it follows that

$$D_1 + 1 = \operatorname{rank}(\Omega_1) \leq s_1 = \operatorname{card}\{\operatorname{Im}(\pi_1^*)\}. \quad \square$$

Remark 9. Theorem 3 provides a link between local observations (i.e., the minimum number of successive values that a node r needs to accumulate to compute the final consensus value of the network) and a global property (i.e., the underlying minimum EEP of the network with respect to node r). Using this theorem, one can directly bound the minimum number of steps for particular nodes in the network without resorting to algebraic numerical manipulations.

Remark 10. Empirical numerical results indicate that the upper bound in Theorem 3 is tight for a wide variety of graphs with the equality holding in most cases (Martini et al., in press). We are currently investigating under what conditions these two quantities are equal.

Example 4. As shown through numerical computations in Example 1, the minimum number of steps for nodes 1, 2 and 3 is 8, while nodes 5 and 6 require 12 steps. Fig. 4 shows that the mEEP with respect to node 1 has 4 cells while the mEEP for node 5 has 6 cells. In this case, the number of cells in the corresponding minimum EEPs coincides with the numerical results in Example 1.

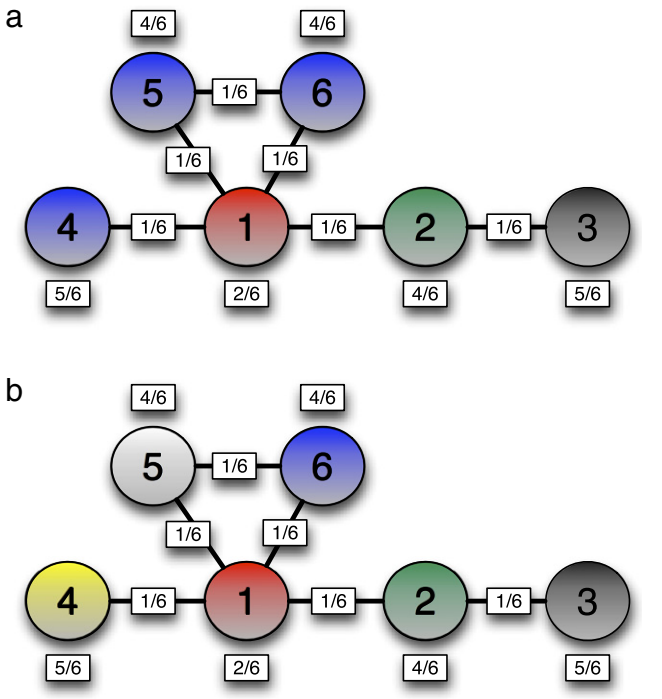

Fig. 4. Minimum EEP of the graph in Fig. 1 and Example 1 with respect to: (a) node 1 (4 cells) and (b) node 5 (6 cells). Different colours correspond to different cells (colour online).

6. Conclusion

This paper formulates and analyses the decentralised minimum time consensus problem. In contrast to other tools in the literature, our algorithm computes the final consensus value from the history of any node in a completely decentralised manner. The necessary information for any node is its own history and is therefore exclusively local, as the algorithm does not require any global knowledge about the network, such as the total number of nodes in the system, information about the neighbourhood of the node, or specific edge weights. After characterising the minimum number of steps required for any given node to compute the final consensus value, we provided algebraic, graph-theoretical and locally informative interpretations of the minimum number of steps.

There are a number of interesting directions for future research in terms of network design. For instance, we are currently working on the problem of computing a minimum EEP with respect to a node in polynomial time. Also it is important to mention that the EEP-based results provided here for undirected graphs can be extended to directed graphs at the price of a more elaborate exposition using graph automorphisms. We plan to extend this work to address the following questions: (a) Given a constraint on the number of edges in the network, what are the network structures that maximise the number of shared eigenvalues between Laplacian and grounded Laplacian? (b) Given a specific network topology, how can we choose the weights of the Laplacian matrix to minimise the minimum number of steps that a chosen node requires to compute the final consensus value? (c) How do robustness aspects such as node or edge failures affect the minimum number of steps in a consensus network?

In terms of applying the proposed minimum-time consensus algorithm to real systems, numerical issues (typically related to the computation of the rank) arise when applying the proposed algorithms to large-scale systems. We are currently investigating

the use of state-of-the-art methods to compute the rank and are also studying the use of graph theoretical means to characterise the minimum number of steps so as to avoid such pervasive numerical issues.

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