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# Finite Horizon LQR Control With Limited Controller-System Communication

Ling Shi, Ye Yuan, and Jiming Chen

***Abstract*—We consider finite-horizon LQR control with limited controller-system communication. Within a time-horizon $T$, the controller can only communicate with the system $d < T$ times. We present an explicit expression on the optimal control data schedule for unstable first-order systems and a class of higher-order systems. We also discuss when such a control data schedule remains optimal (or is near optimal) for general systems.**

***Index Terms*—Control data scheduling, LQR control, optimization, Ricatti equation.**

## I. Introduction

Networked control systems have gained much interest in the past decade thanks to the recent advances in network infrastructure, communication architecture and computer technology [1]. Control over communication networks can reduce system wiring and hence reduce the operational cost, and make efficient use of shared resources such as network bandwidth, central control unit, etc. However, new issues arise when the control loop is closed via a network. For example, network induced delays and data packet drops may severely degrade system performance and may even cause instability [2].

Sinopoli *et al.* [3] took a look at how packet loss affects state estimation. They showed that there exists a certain threshold of the packet arrival rate below which, $\mathbb{E}[P_k]$, the expected value of the error covariance matrix, becomes unbounded as time goes to infinity. They also provided lower and upper bounds of the threshold value. The authors extended their result from estimation to LQG control in [4] where stability region of packet arrival rates were provided. Gupta *et al.* [5] considered LQG control over a packet-dropping network. By using a separation principle, they decomposed the problem into a standard LQR controller design, together with an optimal encoder-decoder design for propagating and using the information across the packet-dropping network. Their proposed encoder-decoder was proved to be optimal among all encoder-decoders.

In many networked control applications, network resources such as the available bandwidth and computational unit have to be shared by many systems accessing the network at the same time, thus proper scheduling schemes are often required to guarantee certain desired properties of those systems. For example, Walsh *et al.* [6], [7] studied the problem of when to schedule which process to access the network so that all processes remain stable.

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Control/sensor data scheduling is in general a difficult and challenging task and most existing results rely heavily on relaxation techniques and often only generate suboptimal schedules [8]. Limiting the control actions often yields a different mathematical formulation than limiting the available sensor observations, although in many cases, control and estimation are dual concepts of each other, e.g., controllability/observability. For example, consider the problem of LQG control over a packet-dropping network [4]. Assume there is no acknowledgement on whether the control or the sensor data arrive successfully. Then limiting the sensor measurements only affects the state estimation error covariance, while limiting the control actions affects both the state estimation error and the system performance. In light of the difficulty tackling scheduling both the control data and sensor measurements simultaneously, most existing works looked at either limited sensor observations or limited control actions.

The early work by Kushner [9] considered optimal timing of sensor observations for a scalar discrete-time linear system with unknown initial state. By using the tools of optimal stochastic control theory, the optimal timing of the available sensor measurements were determined which minimizes the expected value of a cost function that is quadratic in the control and the terminal position error. Following [9], Sano and Terao [10] investigated optimal timing of sensor observations for a continuous-time linear system and obtained analytic solutions for a class of special scalar systems. Skafidas and Nerode [11] also considered optimal timing of sensor observations for LQG control, and they showed that the cost function is only a function of the measurement times rather than the measurements themselves, which allows the optimization to be performed offline. No closed-form solution, however, was given but numerical examples were provided to demonstrate how such an optimization can be carried out. Instead of taken the number of available sensor measurements as a hard constraint, Molin and Hirche [12] looked at a cost function which consists of the classic LQG quadratic cost and a weighting factor on each sensor measurement. The latter one is treated as a penalty cost for communication between the sensor and the remote controller. Optimal scheduling law can be obtained by using dynamic programming algorithm. Gupta *et al.* [13] proposed a stochastic sensor scheduling scheme and provided the optimal probability distribution over the sensors to be selected. Sandberg *et al.* [14] considered estimation over a heterogeneous sensor network. Two types of sensors were investigated: the first type has low-quality measurement but small processing delay, while the second type has high-quality measurement but large processing delay. Using a time-periodic Kalman filter, they showed how to find an optimal schedule of the sensor communication. Savage and La Scala [15] considered the problem of optimal measurement scheduling for scalar systems that minimizes the terminal error. Yang and Shi [16] also considered optimal measurement scheduling for remote state estimation of scalar systems, but focused on minimizing the average estimation error over a finite time horizon. Under some mild assumptions, the authors showed that under an optimal sensor data schedule, the sensor-to-estimator communication instances should be separated as uniformly as possible.

Imer and Başar [17] considered optimal LQG control of a scalar system with limited control actions. They showed that the optimal control is a threshold policy on the best estimate of the system state which can be generated by a Kalman filter. Bommannavar and Başar [18] considered optimal LQG control of a class of higher-order systems with limited control actions. The cost function does not penalize the control actions and it was also shown that the optimal control is a threshold policy, and the optimal thresholds can be obtained numerically. Similar problems with limited control actions were investigated by Shemonski [19]. More related topics and results can be found from the references in the aforementioned existing work.

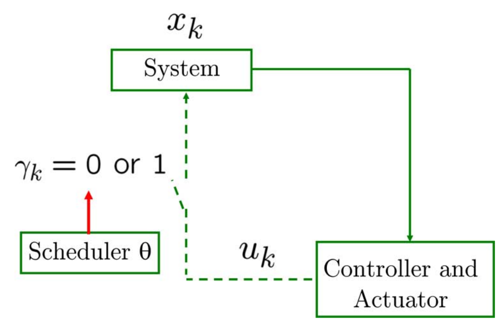


Fig. 1. Optimal scheduling of control data.

In this paper, we consider finite-horizon LQR control under limited communication between the controller and the system. The main contributions of this paper are summarized as follows.

1) For unstable first-order systems and a class of higher-order systems for which condition (8) holds, we provide a closed-form expression for an optimal control data schedule, which is in contrast to most available works in literature where only numerical solutions were provided or only scalar systems were considered.
2) For a general higher-order system, we discuss the cases when the control data schedule in the first part remains (or is near) optimal.

The remainder of the paper is organized as follows. Section II gives the mathematical problem setup. Section III presents the main result of this paper. Some discussions are included in Section IV and concluding remarks are provided in the end.

*Notations:* $\mathbb{Z}$ is the set of non-negative integers. $k \in \mathbb{Z}$ is the time index. $\mathbb{N}$ is the set of natural numbers. $\mathbb{R}^n$ is the $n$-dimensional Euclidean space. $\mathbb{S}_+^n$ is the set of $n$ by $n$ positive definite matrices. When $X \in \mathbb{S}_+^n$, we simply write $X > 0$. $X > Y$ means $X - Y > 0$. For functions $f_1, f_2, f : \mathbb{S}_+^n \to \mathbb{S}_+^n$, $f_1 \circ f_2(X) = f_1 f_2(X) \triangleq f_1(f_2(X))$, $f^t \triangleq \underbrace{f \cdots f}_{t \text{ times}}(X)$ and $f^0(X) \triangleq X$. $\mathbb{E}[\cdot]$ denotes the expectation of a random variable. $\mathrm{Tr}(\cdot)$ denotes the trace of a matrix.

## II. Problem Setup

Consider the following discrete linear time-invariant system in Fig. 1. The system dynamics is given by

$$x_{k+1} = Ax_k + \gamma_k B u_k \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the system state at time $k$, $u_k \in \mathbb{R}^m$ is the control input at time $k$, and $\gamma_k \in \{0, 1\}$ is the decision variable at the controller (actuator) whether it sends $u_k$ to the system or not. We assume $A$ is unstable with $(A, B)$ being controllable. This decision-making at the controller (actuator) whether to send $u_k$ or not arises for example in the following application scenarios.

1) The controller is a shared computation unit and is not dedicated to work for the system.
2) The communication bandwidth between the controller and the system is finite.
3) The actuator is time-shared by a few other systems and can only be used for system (1) for a limited number.
4) There is a critical cost (e.g., energy) associated with each use of the actuator.

Consider any initial condition $x_0 \in \mathbb{R}^n$ which is known to the controller (hence all future state $x_k, k \geq 1$ are known to the controller).

The controller computes the optimal linear control law $u_k = -L_k x_k$ such that the following cost function:

$$\sum_{k=0}^{T-1} \left(x_k' Q x_k + \gamma_k u_k' R u_k\right) + x_T' Q x_T$$

is minimized, where $Q \in \mathbb{S}_+^n$, $R \in \mathbb{S}_+^m$, and $T \in \mathbb{N}$ is the time-horizon of interest. We assume that $(A, \sqrt{Q})$ is observable.

Assume the controller can communicate with the system for $d$ ($d \in \mathbb{N} \text{ and } d < T$) times, which is equivalent to say that

$$\sum_{k=0}^{T-1} \gamma_k = d. \tag{2}$$

A schedule $\theta$ at the controller consists of a sequence of binary-valued variables $\{\gamma_k(\theta) : k = 0, \ldots, T-1\}$. Apparently, $\Theta$, the set of all such schedules, contains $\binom{T}{d}$ elements.

For a given schedule $\theta$, define $J(\theta)$ as

$$J(\theta) \triangleq \min_{u_k = -L_k x_k} \sum_{k=0}^{T-1} \left(x_k' Q x_k + \gamma_k(\theta) u_k' R u_k\right) + x_T' Q x_T. \tag{3}$$

In this technical note, we are interested in the following problem.

*Problem 2.1:*

$$\min_{\theta \in \Theta} J(\theta)$$
$$\text{s.t.} \sum_{k=0}^{T-1} \gamma_k(\theta) = d. \tag{4}$$

In other words, we wish to find a controller-system communication schedule $\theta$ which satisfies (4) and minimizes the cost function $J(\theta)$.

For simplicity, we write $J(\theta)$ and $\gamma_k(\theta)$ as $J$ and $\gamma_k$ when the underlying schedule $\theta$ is clear from the context. $\theta$ is *feasible* if it satisfies (4) and is *optimal* if it is feasible and for any other feasible schedule $\hat{\theta}$, $J(\theta) \le J(\hat{\theta})$.

## III. Main Result

In this section, we present the main result of this note. First we transform Problem 2.1 into a different form that is easier to handle. We will make use of the following proposition. The proof is simply based on dynamic programming and is omitted.

*Proposition 3.1:* For a given schedule $\theta$, the control law $u_k$ that minimizes $J(\theta)$ defined in (3) is given by

$$u_k = -[B' S_{k+1} B + R]^{-1} B' S_{k+1} A x_k$$

where $S_k$ is given recursively by

$$S_k = A' S_{k+1} A + Q - A' S_{k+1} B [B' S_{k+1} B + R]^{-1} B' S_{k+1} A$$

if $\gamma_k = 1$ and

$$S_k = A' S_{k+1} A + Q$$

if $\gamma_k = 0$. The above recursion starts from $S_T = Q$. Furthermore, the cost function $J(\theta)$ is given by

$$J(\theta) = x_0' S_0 x_0.$$

To facilitate the analysis later, define the functions $h, g : \mathbb{S}_+^n \to \mathbb{S}_+^n$ as

$$h(X) \triangleq A' X A + Q \tag{5}$$
$$g(X) \triangleq A' X A + Q - A' X B [B' X B + R]^{-1} B' X A. \tag{6}$$

Since $(A, B)$ is controllable and $(A, \sqrt{Q})$ is observable, from the analysis of standard *algebraic Riccati equation* (e.g., [20, Proposition 4.4.1]), the equation $g(X) = X$ has a unique positive definite solution $P > 0$, i.e.,

$$P = h(P) - A' P B [B' P B + R]^{-1} B' P A. \tag{7}$$

Furthermore for all initial condition $P_0 \ge 0$, the recursion $P_{k+1} = g(P_k)$ converges to $P$, i.e., $\lim_{k \to \infty} P_k = P$. From Proposition 3.1, Problem 2.1 is equivalent to the following:

*Problem 3.2:*

$$\min_{\theta \in \Theta} x_0' S_0 x_0$$
$$\text{s.t. } S_T = Q$$
$$S_k = \begin{cases} g(S_{k+1}), & \text{if } \gamma_k = 1 \\ h(S_{k+1}), & \text{if } \gamma_k = 0. \end{cases}$$
$$\sum_{k=0}^{T-1} \gamma_k = d.$$

Note that an optimal schedule to Problem 3.2 in general may depend on the initial state $x_0$. However, as we shall see from the following main result of this note, if (8) or the conditions of Proposition 3.7 holds, then the greedy, $x_0$-independent schedule given by (9) is optimal.

*Theorem 3.3:* If $A, B, Q, R$ satisfy

$$gh(X) \le hg(X), \quad \forall X > 0, \tag{8}$$

then an optimal schedule $\theta^\star$ to Problem 3.2 is given in terms of $\gamma_k(\theta^\star)$ as follows:

$$\gamma_k(\theta^\star) = 1 \forall 0 \le k \le d-1 \quad \text{and} \quad \gamma_k(\theta^\star) = 0 \forall k \ge d. \tag{9}$$

We need the following tool to prove the theorem. From [3], for any $Y \ge X > 0$, $h$ and $g$ satisfy $h(X) \le h(Y)$, $g(X) \le g(Y)$ and $g(X) \le h(X)$. The following property on $h$ and $g$ is essential to prove Theorem 3.3.

*Lemma 3.4:* If (8) holds, then for any $i, t \in \mathbb{Z}$ and $X \ge 0$

$$g^t h^i(X) \le h^i g^t(X). \tag{10}$$

*Proof:* First note that (10) holds trivially when $t = 0$. When $t = 1$, by (8), we have

$$\begin{aligned} gh^i(X) &= gh\left(h^{i-1}(X)\right) \\ &\le hg\left(h^{i-1}(X)\right) \\ &\le h^2\left(gh^{i-2}(X)\right) \\ &\vdots \\ &\le h^i g(X). \end{aligned}$$

Now assume (10) holds for any $t \le k$. Then

$$\begin{aligned} g^{k+1}h^i(X) &= g\left(g^k h^i(X)\right) \\ &\le gh^i g^k(X) \\ &\le h^i g g^k(X) \\ &= h^i g^{k+1}(X). \end{aligned}$$

Therefore, by induction, (10) holds for any $i, t \in \mathbb{Z}$. Also note that if $t = 0$ or $i = 0$, (10) becomes an equality. ∎

We are now ready to prove Theorem 3.3.

*Proof to Theorem 3.3:* We first prove $\theta^\star$ defined by (9) is optimal by showing that for any $\theta \in \Theta$, $S_0(\theta^\star) \le S_0(\theta)$. Notice that from this we obtain

$$J(\theta^\star) = x_0' S_0(\theta^\star) x_0 \le x_0' S_0(\theta) x_0 = J(\theta).$$

Since $S_k$ is given by

$$S_k = \begin{cases} g(S_{k+1}), & \text{if } \gamma_k = 1 \\ h(S_{k+1}), & \text{if } \gamma_k = 0 \end{cases}$$

for any feasible schedule $\theta$, $S_0(\theta)$ can be written as

$$S_0(\theta) = h^{j_1} g^{t_1} h^{j_2} g^{t_2} \cdots h^{j_d} g^{t_d} h^{j_{d+1}}(Q) \tag{11}$$

for some $t_i \in \mathbb{Z}, i = 1, \ldots, d$ and some $j_i \in \mathbb{Z}, i = 1, \ldots, d+1$ which satisfy

$$\sum_{i=1}^{d} t_i = d, \quad \sum_{i=1}^{d+1} j_i = T - d.$$

Since $S_0(\theta^\star) = g^d h^{T-d}(Q)$, to show $S_0(\theta^\star) \le S_0(\theta)$ we need to establish that

$$g^d h^{T-d}(Q) \le h^{j_1} g^{t_1} h^{j_2} g^{t_2} \cdots h^{j_d} g^{t_d} h^{j_{d+1}}(Q). \tag{12}$$

By (10) and the fact that both $g$ and $h$ are increasing functions, we have that

$$\begin{aligned} &h^{j_1} g^{t_1} h^{j_2} g^{t_2} \cdots h^{j_d} g^{t_d} h^{j_{d+1}}(Q) \\ &\qquad \ge g^{t_1} h^{j_1} h^{j_2} g^{t_2} \cdots h^{j_d} g^{t_d} h^{j_{d+1}}(Q). \end{aligned}$$

By (10), again we have that

$$\begin{aligned} &h^{j_1} g^{t_1} h^{j_2} g^{t_2} \cdots h^{j_d} g^{t_d} h^{j_{d+1}}(Q) \\ &\quad \ge g^{t_1} h^{j_1} h^{j_2} g^{t_2} \cdots h^{j_d} g^{t_d} h^{j_{d+1}}(Q) \\ &\quad \ge g^{t_1+t_2} h^{j_1} h^{j_2} \cdots h^{j_d} g^{t_d} h^{j_{d+1}}(Q). \end{aligned}$$

Iterating the same argument we arrive at (12). ∎

*Remark 3.5:* A dual result (for scalar systems only) in the context of state estimation is obtained in [15], where the sensor takes measurement during *the last $d$ time steps* to minimize the terminal error covariance. Theorem 3.3 says that to minimize the LQR cost, the control steps should be applied during *the first $d$ time steps*.

*Remark 3.6:* The schedule (9) is also the optimal open-loop scheduling policy in [17] (with $R \to 0$), where a scalar system is considered. This is reflected in the first part of Proposition 3.7.

The next result characterizes a class of systems for which the schedule $\theta^\star$ presented in Theorem 3.3 is optimal.

*Proposition 3.7:* For the following two special systems, condition (8) holds if:

1) $n = 1$ and $|A| \ge 1$, i.e., unstable first-order systems;
2) $n \ge 2$ and the matrices $B, Q, R$ satisfy $(BR^{-1}B')^{-1} \le Q$.

*Proof:* 1) Straightforward calculation shows that (8) holds when $n = 1$ and $|A| \ge 1$. 2) From the Matrix Inversion Lemma, for any $X > 0$, $g(X)$ can be written as

$$g(X) = A'[X^{-1} + BR^{-1}B']^{-1} A + Q.$$

Thus, if $(BR^{-1}B')^{-1} \le Q$, then

$$\begin{aligned} gh(X) &= A'\left[(h(X))^{-1} + BR^{-1}B'\right]^{-1} A + Q \\ &\le A'(BR^{-1}B')^{-1} A + Q \le A'QA + Q \\ &\le A'\left[A'(X^{-1} + BR^{-1}B')^{-1} A + Q\right] A + Q \\ &= hg(X). \end{aligned}$$

∎

Proposition 3.7 presents two classes of systems in which the schedule (9) is optimal. The first class consists of unstable first-order systems. The second class consists of a special higher-order systems, where the physical meaning of $(BR^{-1}B')^{-1} \le Q$ is that the control input weighting is smaller than that of the state weighting. In this case, since the control is relatively "cheaper", it is "wise" to use large control to bring down the state at the beginning.

## IV. Discussions

### A. A General Constraint

The constraint (4) can have a general form as

$$\sum_{k=0}^{T-1} \gamma_k(\theta) \le d$$

i.e., the controller is allowed to communicate with the system for at most $d$ times. The optimal schedule is, however, the same as that of Problem 2.1. A quick proof to this runs as follows.

Suppose $\theta^\star$ is optimal with $\sum_{k=0}^{T-1}\gamma_k(\theta) < d$. Let $k_1$ be a time such that under $\theta^\star$, $\gamma_{k_1}(\theta^\star) = 0$. Let $\hat{\theta}$ be identical to $\theta^\star$ except that $\gamma_{k_1}(\hat{\theta}) = 1$. Notice that

$$\sum_{k=0}^{T-1}\gamma_k(\hat{\theta}) = \sum_{k=0}^{T-1}\gamma_k(\theta^\star) + 1 \le d$$

thus $\hat{\theta}$ is still feasible. Now

$$\begin{aligned} J(\theta^\star) &= x_0'\left(f_0\cdots f_{k_1-1}hf_{k_1+1}\cdots f_{T-1}(Q)\right)x_0 \\ &\ge x_0'\left(f_0\cdots f_{k_1-1}gf_{k_1+1}\cdots f_{T-1}(Q)\right)x_0 \\ &= J(\hat{\theta}) \end{aligned}$$

where $f_k = h$ if $\gamma_k(\theta^\star) = 0$ and $f_k = g$ otherwise. Since $h \neq g$, there exists $x_0 \in \mathbb{R}^n$ such that the above inequality becomes strict. Hence we conclude that $\hat{\theta}$ is better than $\theta^\star$. In other words, $\theta^\star$ cannot be optimal. Consequently, an optimal schedule has to satisfy (4).

### B. General Systems

In Proposition 3.7, we see that if $B, Q, R$ satisfy $(BR^{-1}B')^{-1} \le Q$, then the schedule $\theta^\star$ constructed in Theorem 3.3 is optimal. In this section, we consider the case when the aforementioned condition for $B, Q, R$ does not hold. Let $c_i$ be the controllability index for $(A, B)$, i.e., $c_i$ is the smallest integer such that the matrix

$$M_{c_i} \triangleq [B\ AB \cdots A^{c_i-1}B] \tag{13}$$

has rank $n$. One immediately obtains that $1 \le c_i \le n$ since $(A, B)$ is controllable. One also notices that if $B$ is invertible, then $c_i = 1$, and if $B$ is a single column vector, then $c_i = n$.

We focus on the case when $d \ge c_i$. If $d < c_i$, to the best of our knowledge, very little can be said on the optimal schedule. One may invoke computational tools such as enumerating and comparing all possible schedules to obtain the optimal schedule, since in this case $c_i$ is small and consequently $d$ is even smaller.

When $d \ge c_i$, instead of searching for the optimal schedule, which is a challenging task, we provide an upper bound on the performance difference between $\theta^\star$ proposed in (9) with the true optimal schedule. We also show that this upper bound decays to zero exponentially with $d$. First consider controlling system (1) over the time horizon 0 to $c_i$ (assuming $\gamma_k = 1$ for all $k$) with the following cost:

$$J(x_0) = \sum_{k=0}^{c_i-1}\left[x_k'Qx_k + u_k'Ru_k\right] + x_{c_i}'Xx_{c_i}.$$

The minimum cost, from Proposition 3.1, is given by

$$J^\star(x_0) = x_0'g^{c_i}(X)x_0.$$

Let $\mathbf{u_{c_i}} = [u_{c_i-1}\ u_{c_i-2}\cdots u_0]$. Then we can write $x_{c_i}$ as

$$x_{c_i} = A^{c_i}x_0 + M_{c_i}\mathbf{u_{c_i}}.$$

By letting $\mathbf{u_{c_i}} = -M_{c_i}'(M_{c_i}M_{c_i}')^{-1}A^{c_i}x_0$, the terminal state $x_{c_i} = 0$. This choice of $\mathbf{u_{c_i}}$ is valid as $M_{c_i}$ has rank $n$ and consequently $M_{c_i}M_{c_i}'$ is invertible. Notice that $x_k$ and $u_k$ can be written as $x_k = Y_kx_0$ and $u_k = Z_kx_0$ for some constant matrices $Y_k$ and $Z_k$. Therefore

$$\begin{aligned} J^\star(x_0) &= x_0'g^{c_i}(X)x_0 \\ &\le \sum_{k=0}^{c_i-1}\left(x_k'Qx_k + u_k'Ru_k\right) + x_{c_i}'Xx_{c_i} \\ &= x_0'Mx_0 \end{aligned}$$

where

$$M \triangleq \sum_{k=0}^{c_i-1}Y_k'QY_k + Z_k'RZ_k. \tag{14}$$

Note that $x_0$ can be any arbitrary vector in $\mathbb{R}^n$, and as a result, $g^{c_i}(X) \le M$. We summarize the above observation in the following lemma.

*Lemma 4.1:* Let $M$ be defined as in (14). Then $g^{c_i}(X) \le M$ for any $X \ge 0$.

Let $\theta^\star$ now denote the true optimal schedule to Problem 3.2. Notice that such an optimal $\theta^\star$ must exist since the number of all feasible schedules is finite. Let $\theta$ be defined according to (9), i.e., $\gamma_k(\theta) = 1, k = 0, \ldots, d-1$ and $\gamma_k(\theta) = 0, k \ge d$. Then we have the following result.

*Theorem 4.2:* If $d \ge c_i$, then

$$|J(\theta) - J(\theta^\star)| \le x_0'\left[g^{d-c_i}(M) - g^T(Q)\right]x_0 \tag{15}$$

where $M$ is defined in (14).

*Proof:* From Lemma 4.1

$$\begin{aligned} J(\theta) &= x_0'g^dh^{T-d}(Q)x_0 \\ &= x_0'g^{d-c_i}\left(g^{c_i}h^{T-d}(Q)\right)x_0 \\ &\le x_0'g^{d-c_i}(M)x_0. \end{aligned}$$

On the other hand

$$J(\theta^\star) = x_0'\left(f_0\circ f_1\circ\cdots\circ f_{T-1}(Q)\right)x_0 \ge x_0'g^T(Q)x_0 \tag{16}$$

where $f_k = g$ if $\gamma_k(\theta^\star) = 1$ and $f_k = h$, otherwise. By the optimality of $\theta^\star$, we have

$$x_0'g^T(Q)x_0 \le J(\theta^\star) \le J(\theta) \le x_0'g^{d-c_i}(M)x_0$$

which proves the theorem. ■

*Remark 4.3:* From Proposition 4.4.1 in [20] and its proof, $\lim_{k\to\infty}g^k(P_0) = P$, $\forall P_0 \ge 0$ and the convergence of $g^k(P_0)$ to $P$ is exponential in $k$. Furthermore, the convergence rate is given by $\rho$, the spectral radius of the closed-loop stable matrix

$$A + BL = A - B(B'PB + R)B'PA.$$

In other words, there exists a constant $\alpha(P_0)$ such that

$$-\alpha(P_0)\rho^kI \le g^k(P_0) - P \le \alpha(P_0)\rho^kI.$$

From this fact, we obtain the following:

$$\begin{aligned}|J(\theta) - J(\theta^\star)| &\leq x_0' \left[g^{d-c_i}(M) - g^T(Q)\right] x_0 \\ &= x_0' \left[g^{d-c_i}(M) - P - \left(g^T(Q) - P\right)\right] x_0 \\ &\leq \left|x_0' \left(g^{d-c_i}(M) - P\right) x_0\right| \\ &\quad + \left|x_0' \left(g^T(Q) - P\right) x_0\right| \\ &\leq \alpha_1(x_0, M)\rho^{d-c_i} + \alpha_2(x_0, Q)\rho^T \\ &= \left[\alpha_1(x_0, M)\rho^{d-c_i} + \alpha_2(x_0, Q)\rho^{T-d}\right] \rho^d \\ &\leq \left[\alpha_1(x_0, M)\rho^{-c_i} + \alpha_2(x_0, Q)\right] \rho^d\end{aligned}$$

where $\alpha_1(x_0, M)$ and $\alpha_2(x_0, Q)$ are two positive constants and are independent of $\rho$. Therefore, when $d \gg c_i$, we have

$$|J(\theta) - J(\theta^\star)| \approx 0$$

i.e., the constructed $\theta$ is close to the real optimal $\theta^\star$. Notice that $d$ can still be significantly less than $T$ as the following example shows.

*Example 4.4:* Consider the following third-order system:

$$A = \begin{bmatrix} 2 & 0.5 & 0.5 \\ 0 & 3 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and $Q = 0.5 * I_3, R = 1, T = 500$. For this system, $c_i = n = 3$. We consider the following different values of $d$.

$d < 3$: In this case, there must be at least 166 consecutive times during which no control is applied. Under any feasible schedule, the cost $J(\theta)$ quickly diverges during these 166 time steps as the system is unstable and is not controllable with less than three control inputs.

$d = 5$: Following the procedure of calculating $Y_k$ and $Z_k$ prior to equation (18), we get

$$M = 10^4 * \begin{bmatrix} 3.2075 & -5.5285 & -0.2564 \\ -5.5285 & 9.5801 & 0.4519 \\ -0.2564 & 0.4519 & 0.0226 \end{bmatrix}.$$

According to Theorem 4.2, we find that

$$J(\theta^\star) \geq x_0' g^{500}(Q) x_0 = 55.73$$

and

$$|J(\theta) - J(\theta^\star)| \leq x_0' \left[g^2(M) - g^{500}(Q)\right] x_0 = 15.46$$

i.e., $\theta$ differs from $\theta^\star$ by *at most* 27%. One may notice that the controls are only applied in the first five steps while the system is running open loop during the remaining 495 time steps. This may seem to be even worse than the $d < 3$ case. The truth is that with $d = 5$, the optimal control $u_k$ given by Proposition 3.1 drives $x_k$ arbitrarily close to the origin at $k = 5$ and $x_k$ remains close to the origin afterwards. On the other hand, when $d < 3$, $x_k$ *cannot be made arbitrarily close to the origin* in any $k$ with less than three controls, hence it diverges.

$d = 10$: This time

$$|J(\theta) - J(\theta^\star)| \leq x_0' \left[g^7(M) - g^{500}(Q)\right] x_0 = 0.071$$

i.e., $\theta$ differs from $\theta^\star$ by *at most* 0.13%.

$d = 15$: The difference is shown to be

$$|J(\theta) - J(\theta^\star)| \leq x_0' \left[g^{12}(M) - g^{500}(Q)\right] x_0 = 0.0001$$

i.e., the performance between $\theta$ and $\theta^\star$ is *negligible*.

Therefore, even condition (8) may not hold in this example, the proposed schedule (9) has almost the same performance as the true optimal schedule, provided that $d$ is slightly bigger than the controllability index $c_i$. Note that in all three cases, $d \ll T$, and the controller-to-actuator communication rate equals to 1%, 2% and 3%, respectively. We also notice that $|J(\theta) - J(\theta^\star)|$ decays exponentially in $d$, which agrees with the previous analysis.

### C. Two Special Cases

In this section we consider two special cases: 1) when $d$ is fixed but $T \to \infty$ and 2) when both $d$ and $T \to \infty$ but the ratio $(d/T) \to 0$. For both cases, we assume $d > c_i$. We have the following result.

*Proposition 4.5:*

1) When $d$ is fixed and $T \to \infty$,

$$\lim_{T \to \infty} |J(\theta) - J(\theta^\star)| \leq \delta$$

for some constant $\delta$.

2) When both $d$ and $T \to \infty$ with $(d/T) \to 0$,

$$\lim_{T \to \infty} |J(\theta) - J(\theta^\star)| = 0.$$

*Proof:*

1) Taking the limit as $T \to \infty$ on the right-hand-side of (15) and using the fact that $\lim_{T \to \infty} g^T(Q) = P$, one has

$$\lim_{T \to \infty} |J(\theta) - J(\theta^\star)| \leq x_0' \left[g^{d-c_i}(M) - P\right] x_0 \triangleq \delta.$$

2) Clearly as $d \to \infty$, $g^{d-c_i}(M) \to P$, thus

$$\begin{aligned}&\lim_{d,T \to \infty, \frac{d}{T} \to 0} |J(\theta) - J(\theta^\star)| \\ &\leq \lim_{d,T \to \infty, \frac{d}{T} \to 0} x_0' \left[g^{d-c_i}(M) - P\right] x_0 \\ &= 0.\end{aligned}$$

■

In the first case, one sees that as long as $T$ is sufficiently large, the cost function under our proposed schedule will not deviate much from the real optimal schedule. In the second case, when $d$ is also sufficiently large (the duty cycle can be made at the same time arbitrarily close to zero), one sees that our proposed schedule is arbitrarily close to the real optimal schedule.

## V. Conclusion

A finite horizon LQR control where the controller is allowed to communicate with the system $d < T$ times within a time horizon $T$ is considered in this paper. Optimal control data schedule is presented in closed-form for a class of systems and some discussions on the optimal schedule for general systems are presented.

Future work along the line of this work include finding the exact optimal schedule for general higher-order systems and LQR control with output feedback, and considering LQG control data scheduling.

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# Realization of a Special Class of Admittances with One Damper and One Inerter for Mechanical Control

Michael Z. Q. Chen, Kai Wang, Yun Zou, and James Lam

***Abstract*—In this note, we investigate the realization problem of a special class of positive-real admittances, which is common in vehicle suspension designs. The number of inerters and dampers is restricted to one in each case and the number of the springs is arbitrary. To solve the problem, we first convert a previous result by [6] to a more direct form. A necessary and sufficient condition for realizability is then derived and explicit circuit arrangements are provided by assuming that the three-port network consisting of only springs after extracting the damper and the inerter has a well-defined impedance. To remove the assumption on the existence of a well-defined impedance, a condition is established on the topological property of the $n$-port network without a well-defined impedance to obtain an equivalent class of such networks so that the realizability condition is derived with realization. By combining the conditions with and without a well-defined impedance, the final realization result is obtained.**

***Index Terms*—Electric circuits, inerter, mechanical networks, network synthesis, passivity.**

## I. Introduction

Passive network synthesis is a classical subject in electrical circuit theory which experienced a "golden era" for the 1930s–1970s [1], [2], [11], [16]. Despite the relative maturity of the field, certain aspects of passive network synthesis are still incomplete. For example, the only general method for transformerless electrical synthesis by Bott and Duffin [2] appears to be highly non-minimal. However, interest in the field has declined despite the relatively recent development in the design of positive real functions [8], [10], [15], [20].

Recently, a new network element, named *inerter* [4], [19], has been introduced with the property that the (equal and opposite) force applied at the terminals is proportional to the relative acceleration between them. Applications of the inerter to vehicle suspension, motorcycle steering control and vibration absorption have been identified with performance advantages demonstrated (see [4] and references therein). One of the main motivations for the inerter was the synthesis of passive mechanical networks. The inerter completes the analogy between electrical networks and mechanical ones (see [19, Fig. 4]), which makes any passive mechanical network realizable with three kinds of passive elements: inerters, dampers, and springs. However, the number of elements for mechanical networks is much more essential than electrical ones. Therefore, given the existing and potential applications of the inerter, interest in passive network synthesis has been revived [5]–[7], [12], [13]. The need for a renewed attempt on the same subject and its fundamental connection to system theory has also been highlighted by Kalman [14].

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