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Brief paper

Guaranteed cost control of affine nonlinear systems via partition of unity method ☆

Dongfang Han [a,1], Ling Shi [b]

[a] School of Mathematics and Statistics, South-Central University for Nationalities, Wuhan 430074, PR China

[b] Electronic and Computer Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong

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ABSTRACT

We consider the problem of guaranteed cost control (GCC) of affine nonlinear systems in this paper. Firstly, the general affine nonlinear system with the origin being its equilibrium point is represented as a linear-like structure with state-dependent coefficient matrices. Secondly, partition of unity method is used to approximate the coefficient matrices, as a result of which the original affine nonlinear system is equivalently converted into a linear-like system with modeling error. A GCC law is then synthesized based on the equivalent model in the presence of modeling error under certain error condition. The control law ensures that the system under control is asymptotically stable as well as that a given cost function is upper-bounded. A suboptimal GCC law can be obtained via solving an optimization problem in terms of linear matrix inequality (LMI), in stead of state-dependent Riccati equation (SDRE) or Hamilton–Jacobi equations that are usually required in solving nonlinear optimal control problems. Finally, a numerical example is provided to illustrate the validity of the proposed method.

1. Introduction

Control of nonlinear systems has remained an active and difficult issue in control theory. Since there are no uniform methods in dealing with general nonlinear systems, one possible way is to convert nonlinear systems into some linear-like systems using effective approximation methods and then conduct control analysis and synthesis (Hauser, Sastry, & Kokotovic, 1992; Mareels, Penfold, & Evans, 1992; Nesic, Teel, & Kokotovic, 1999; Tang, 2005).

Differential geometry has proved to be powerful in the analysis and design of nonlinear control systems (Isidori, 1995). One may transform nonlinear system models into many kinds of canonical models using differential geometric approaches, then design control laws based on these models (Cheng & Lin, 2002; Isidori, 1995; Marino & Tomei, 1993). Partition of unity is a concept in differential geometry, which is closely related to a group of open covering sets. It has been used to improve the traditional finite element methods in mechanics and engineering (Belytschko, Krongauz, Organ, Fleming, & Krysl, 1996; Melenk & Babuska, 1996). However, there rarely exists published work addressing control problems by using partition of unity method. Partition of unity possesses the properties of boundedness, addition-to-unity and universal existence. It has been proved that certain finite linear combinations of partition of unity are able to approximate arbitrary continuous functions defined in a compact region within any specified accuracy (Wang, Li, & Zhang, 2004), which inspires us that when a nonlinear system is well-approximated using partition of unity method, the remaining control problems become feasible and much easier, see e.g., Han and Wang (2008, 2009) where the problems of GCC for a particular class of nonlinear time-delay systems, and H_∞ control for a class of nonlinear systems have been addressed.

GCC was firstly introduced in adaptive control by Chang and Peng (1972). It ensures system stability as well as an upper bound on a given performance index. Many good results have been available for linear systems, see, e.g., Petersen and McFarlane (1994) and Yu and Chu (1999), and for nonlinear systems, see Chen and Liu (2005), Tang (2005) and Wu and Cai (2006). In Chen and Liu (2005), sufficient conditions for the existence of state-feedback and observer-based output feedback GCC laws for nonlinear time-delay systems were provided in terms of LMIs. Wu and Cai (2006) considered an H_2 guaranteed cost fuzzy control problem for discrete-time uncertain nonlinear systems. While most of these results designed for nonlinear systems are obtained using fuzzy control approach based on certain T–S fuzzy models, neglecting the modeling error between the original system

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E-mail addresses: eastuhan@gmail.com (D. Han), eesling@ust.hk (L. Shi).

[1] Tel.: +86 27 67841900; fax: +86 27 67841900.

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and the fuzzy models. It is well-known that fuzzy models are universal approximators (Takagi & Sugeno, 1985; Wang, 1994). They are able to approximate complex dynamics systems within any specified accuracy. The higher the precision of the model, however, the larger the number of local models in the aggregation. While attempts to maintain a relatively small number of local models inevitably introduce modeling error (Kiriakidis, 1998). Note that the existence of modeling error may be a potential source of instability for control designs which have been based on the assumption that the fuzzy model exactly matches the plant (Cao & Frank, 2000).

In this paper we study the GCC problem for affine nonlinear systems, which cover a large number of mechanical systems (Isidori, 1995; Khalil, 2002). The main contributions of this work are summarized as follows.

(1) We present a novel methodology to investigate control of affine nonlinear systems, which is via approximation using the partition of unity method.
(2) A GCC law is synthesized based on an equivalent transformation of the original nonlinear system in the presence of modeling error using Lyapunov stability theory.
(3) A sufficient condition for the existence of the GCC law is derived and then transferred into the form of LMIs. A suboptimal GCC law can be constructed via solution to a convex optimization problem, which is usually computationally cheap to solve.

The rest of this paper is organized as follows. We present the problem formulation and system model transformation in Section 2. The main results are established in Section 3, followed by Section 4 which shows how some vital issues of the proposed method can be implemented. Section 5 gives a numerical example to illustrate the proposed method. Some final conclusions are provided at the end.

2. System description and preliminaries

Consider the following affine nonlinear system

$$\dot{x} = f(x) + G(x)u, \tag{1}$$

where the state $x \in \overline{U} \subset R^n$, $\overline{U}$ is a compact set in R^n; the input $u \in R^r$; $f : D \to R^n$ is a continuously differentiable map from a domain $D \subset R^n$ into R^n, $G : R^n \to R^{n\times r}$, $G(x) \neq 0, \forall x \in \overline{U}$.

Associated with system (1) is the following cost function

$$J = \int_0^{+\infty} \left[x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)\right] \mathrm{d}t, \tag{2}$$

where Q, R are given positive-definite weighting matrices.

Associated with the cost function (2), a GCC law is defined as follows.

Definition 1. Consider system (1). If there exist a control law $u(t)$ and a scalar J^* such that the closed-loop system is asymptotically stable, and the value of the cost function (2) satisfies $J \le J^*$, then J^* is said to be a guaranteed cost, and the control law $u(t)$ is said to be a GCC law for system (1).

Remark 1. When $G(x)$ reduces to a constant matrix, a special case of (1) was studied in Belta and Habets (2006) where the state x was restricted to a rectangular region of R^n.

Without loss of generality, suppose the origin $x = 0$ is an equilibrium point for the system, i.e., $f(0) = 0$. Then according to Khalil (2002), system (1) can be rewritten as

$$\dot{x} = F(x)x + G(x)u, \tag{3}$$

where $F(x) = (f_{ij}(x))_{n\times n}$ is a state-dependent matrix defined on $\overline{U}$.

Remark 2. Affine nonlinear systems (1) have been investigated widely using extended linearization (Friedland, 1996), also known as apparent linearization (Wernli & Cook, 1975), which is to represent system (1) in the form of (3), and then use the so-called SDRE approach by directly mimicking the LQR formulation. The resulting controller takes the form

$$u(x) = -R^{-1}(x)G^{\mathrm{T}}(x)P(x)x,$$

where $P(x)$ is the unique, symmetric, positive-definite solution to the algebraic SDRE

$$P(x)F(x) + F^{\mathrm{T}}(x)P(x) - P(x)G(x)R^{-1}(x)G^{\mathrm{T}}(x)P(x) + Q(x) = 0, \tag{4}$$

where $R(x)$, $Q(x)$ represents the R, Q in (2), respectively.

Note that (4) is a nonlinear SDRE. It is difficult to solve even if there exists a solution (Shamma & Cloutier, 2003). And global asymptotical stability of the resulting closed-loop system is not guaranteed even after a solution to the SDRE (4) is obtained.

The purpose of this paper is to design a GCC law for system (1) represented in the form of (3), which will be approximated using partition of unity method later. We first introduce some concepts and results related to partition of unity.

Definition 2 (*Boothby, 2003*)**.** Let $\{U_i\}$, $i = 1, \ldots, N$, be an open covering of a compact region $\overline{U}$ of R^n. A C^∞ partition of unity subordinate to the open covering $\{U_i\}$ is a collection of C^∞ functions $\{\alpha_i\}$ defined on the open set $U = \bigcup_{i=1}^N U_i$ with the following properties:

(1) $\alpha_i \ge 0$ on U,
(2) $supp(\alpha_i)$ form a locally finite covering of U, and
(3) $\sum_{i=1}^N \alpha_i(x) = 1$ for every $x \in U$.

where $supp(\alpha_i) := \overline{\{x \in U | \alpha_i \neq 0\}}$ is the closure of the set $\{x \in U | \alpha_i \neq 0\}$.

Lemma 1 (*Boothby, 2003*)**.** *Associated with each open covering $\{U_i\}$ of $\overline{U} \subset R^n$, there exists a C^∞ partition of unity $\{\alpha_i\}$ subordinate to $\{U_i\}$.*

Definition 3 (*Wang et al., 2004*)**.** Let $\mathcal{F}$ be the set of real continuous functions $\tilde{f}$ defined in a compact region $\overline{U}$ of R^n with the condition: $\tilde{f} \in \mathcal{F}$ if and only if there exist a C^∞ partition of unity $\{\alpha_i\}$ subordinate to an open covering $\{U_i\}$ of the compact region $\overline{U}$, $i = 1, \ldots, N$, and N real numbers $\lambda_1, \ldots, \lambda_N$ such that $\tilde{f}(x) = \sum_{i=1}^N \lambda_i \alpha_i(x)$. $\mathcal{F}$ is called an expansion set of partition of unity on $\overline{U}$.

It is easy to validate that the above set $\mathcal{F}$ is a metric space with the sup-metric $d_\infty(f_1, f_2) = \sup_{x\in\overline{U}} |f_1(x) - f_2(x)|$.

Lemma 2 (*Wang et al., 2004*)**.** *For any real continuous function $\xi(x)$ defined in a compact region $\overline{U} \subset R^n$ and arbitrary $\varepsilon > 0$, there exists an $\tilde{f} \in \mathcal{F}$ such that*

$$\sup_{x\in\overline{U}} |\xi(x) - \tilde{f}(x)| < \varepsilon.$$

Note that ε decreases when N increases, and $\varepsilon \to 0$ when $N \to +\infty$. While using Lemma 2, we can first choose an open covering of the compact region, then construct the partition of unity $\{\alpha_i\}$, thus we obtain the expression $\tilde{f}(x) = \sum_{i=1}^N \lambda_i \alpha_i(x)$. Then we can use the principles of linear estimate to determine the parameters λ_i. We show briefly in Section 4 how to construct an appropriate partition of unity.

Lemma 3 (Wang & Han, 2006). *For m real continuous functions $\xi_i = \xi_i(x)$ defined in a compact domain $\overline{U} \subset R^n$ and m arbitrary scalars $\varepsilon_i > 0\,(1 \le i \le m)$, there exists a partition of unity $\{\alpha_j\}, j = 1, \dots, N$, subordinate to the open covering $\{U_j\}$ of $\overline{U}$ and $m \times N$ reals γ_{ij}, such that*

$$\sup_{x\in\overline{U}} \left| \xi_i(x) - \sum_{i=1}^{N} \gamma_{ik}\alpha_k(x) \right| < \varepsilon_i.$$

Lemma 3 implies that different continuous functions can be approximated using only one partition of unity, and the difference between the approximation terms is just that the coefficients of linear combinations of partition of unity are different. This conclusion helps us reduce the design process while using partition of unity to approximate functions.

The following assumptions are required in this paper.

Assumption 1. The origin is the only equilibrium point of system (1) on $\overline{U}$.

Assumption 2. $f(x)$ is locally Lipschitz for all $x \in \overline{U}$, and every solution of the unforced system $\dot{x} = f(x)$ with initial condition $x(0) \in \overline{U}$ lies entirely in a compact set.

Remark 3. Assumption 2 is a sufficient condition for the global existence and uniqueness of the solution of system (1). Another common yet conservative condition is that $f(x)$ is globally Lipschitz. Assumption 2 relaxes the requirement on $f(x)$ at the expense of having to know more about the solution of the system. The trick in checking this assumption that every solution lies in a compact set is that one can analyze the nature of the differential equation without actually solving it (Khalil, 2002, Thm. 3.3), e.g., we can justify, without much difficulty, that each state of the Example 1 evolves within some bounded compact set, respectively. If Assumption 2 is satisfied, then no finite escape phenomenon will occur for system (1).

The above two assumptions imply that if system (1) is asymptotically stable, all its state will converge to zero when time goes to infinity.

Assumption 3. $F(x) = (f_{ij}(x))_{n\times n}, G(x) = (g_{il}(x))_{n\times l}$ are continuous on $\bigcup_{j=1}^{N} U_j$.

For every continuous function $f_{ij}(x)$ or $g_{il}(x),\ i, j = 1, \dots, n;\ l = 1, \dots, r$, we know from Lemma 3 that there exist an open covering $\{U_i\}$ of $\overline{U} \subset R^n$ and a partition of unity $\{\alpha_k\}$ subordinate to it, such that

$$f_{ij}(x) = \sum_{k=1}^{N} \lambda_{ij}^{k}\alpha_k(x) + e_{ij}(x), \tag{5a}$$

$$g_{il}(x) = \sum_{k=1}^{N} \overline{\lambda}_{ij}^{k}\alpha_k(x) + \overline{e}_{ij}(x), \tag{5b}$$

where $e_{ij}(x),\ \overline{e}_{ij}(x)$ are approximation errors, satisfying

$$\sup_{x\in\overline{U}} |e_{ij}(x)| < \varepsilon_{ij}, \qquad \sup_{x\in\overline{U}} |\overline{e}_{ij}(x)| < \delta_{il},$$

where $\varepsilon_{ij} > 0,\ \delta_{il} > 0$.

Now we are going to convert system (3) into a mathematically equivalent linear-like system with modeling error. Let us choose $\lambda_{ij}^{k} = f_{ij}(x^k), \overline{\lambda}_{ij}^{k} = g_{il}(x^k)$ in (5), where x^k is a group of sample values subordinate to the open covering $\{U_k\}$, respectively, $x^k = (x^{k1}, \dots, x^{kn})^{\mathrm{T}},\ k = 1, \dots, N;\ i, j = 1, \dots, n;\ l = 1, \dots, r$. Applying (5), we have

$$F(x) = (f_{ij}(x))_{n\times n} = \left(\sum_{k=1}^{N} \lambda_{ij}^{k}\alpha_k(x) + e_{ij}(x) \right)_{n\times n} = \sum_{k=1}^{N} \alpha_k(x)A_k + (e_{ij}(x))_{n\times n},$$

where $A_k = (\lambda_{ij}^{k})_{n\times n}$. Likewise, we also have

$$G(x) = (g_{il}(x))_{n\times r} = \sum_{k=1}^{N} \alpha_k(x)B_k + (\overline{e}_{il}(x))_{n\times r},$$

where $B_k = (\overline{\lambda}_{il}^{k})_{n\times r}$. Then system (3) can be represented as

$$\dot{x} = \sum_{k=1}^{N} \alpha_k(x)[A_k x + B_k u] + e, \tag{6}$$

where the modeling error $e = (e_{ij}(x))_{n\times n}x + (\overline{e}_{il}(x))_{n\times r}u$.

Assumption 4. $(e_{ij}(x))_{n\times n},\ \ (\overline{e}_{il}(x))_{n\times r}$ are norm-bounded, i.e., there exist finite real constants a, b such that

$$\|(e_{ij}(x))_{n\times n}\| \le a, \qquad \|(\overline{e}_{il}(x))_{n\times r}\| \le b. \tag{7}$$

Remark 4. The above assumption is reasonable, since we consider in a compact region the approximation of continuous functions. The approximation errors will get smaller if the number of open coverings gets larger.

3. Main results

The design scheme of a GCC law becomes standard after system (3) is represented as (6). We propose the following state feedback control law

$$u = \sum_{j=1}^{N} \alpha_j(x)K_j x, \tag{8}$$

where the gain K_j is to be determined by (9).

Theorem 1. *Consider system (3) satisfying Assumptions 1 and 2. If there exist scalars $\varepsilon_1 > 0, \varepsilon_2 > 0$, a common positive-definite matrix P, and matrices K_j, such that the following matrix inequalities are satisfied for $1 \le i, j \le N$,*

$$(A_i + B_iK_j)^{\mathrm{T}}P + P(A_i + B_iK_j) + (\varepsilon_1 + \varepsilon_2)P^2 + \frac{a^2}{\varepsilon_1}I + \frac{b^2}{\varepsilon_2}K_j^{\mathrm{T}}K_j + Q + K_j^{\mathrm{T}}RK_j \le 0, \tag{9}$$

then the control law (8) is a GCC control law for system (3). Moreover, the cost function (2) satisfies

$$J \le x^{\mathrm{T}}(0)Px(0).$$

Proof. Choose the Lyapunov function candidate as $V(x) = x^{\mathrm{T}}Px$. Taking the time derivative of $V(x)$ along the trajectories of the closed-loop system (6) yields

$$\dot{V}(x) = \dot{x}^{\mathrm{T}}Px + x^{\mathrm{T}}P\dot{x} = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i\alpha_j x^{\mathrm{T}}[(A_i + B_iK_j)^{\mathrm{T}}P + P(A_i + B_iK_j)]x + e^{\mathrm{T}}Px + x^{\mathrm{T}}Pe, \tag{10}$$

where $e = \left[\left(e_{ij}(x)\right)_{n\times n} + \sum_{j=1}^{N} \alpha_j \left(\overline{e}_{il}(x)\right)_{n\times r} K_j \right] x$.

Note the fact that for compatible vectors X, Y and arbitrary $\varepsilon > 0, X^{\mathrm{T}}Y + Y^{\mathrm{T}}X \leq \frac{1}{\varepsilon}X^{\mathrm{T}}X + \varepsilon Y^{\mathrm{T}}Y$. From (7), we have

$$\begin{aligned} e^{\mathrm{T}}Px + x^{\mathrm{T}}Pe &= x^{\mathrm{T}}(e_{ij}(x))^{\mathrm{T}}_{n\times n}Px + x^{\mathrm{T}}P(e_{ij}(x))_{n\times n}x \\ &\quad + \sum_{j=1}^{N}\alpha_j x^{\mathrm{T}}K_j^{\mathrm{T}}(\bar{e}_{il}(x))^{\mathrm{T}}_{n\times r}Px + \sum_{j=1}^{N}\alpha_j x^{\mathrm{T}}P(\bar{e}_{il}(x))_{n\times r}K_j x \\ &\leq (\varepsilon_1+\varepsilon_2)x^{\mathrm{T}}P^2x + \frac{a^2}{\varepsilon_1}x^{\mathrm{T}}x + \sum_{j=1}^{N}\alpha_j\frac{b^2}{\varepsilon_2}x^{\mathrm{T}}K_j^{\mathrm{T}}K_j x. \end{aligned} \tag{11}$$

Substituting (11) into (10), we obtain

$$\begin{aligned} \dot{V}(x(t)) &\leq \sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\alpha_j x^{\mathrm{T}}\Bigg[(A_i+B_iK_j)^{\mathrm{T}}P + P(A_i+B_iK_j) \\ &\qquad + (\varepsilon_1+\varepsilon_2)P^2 + \frac{a^2}{\varepsilon_1}I + \frac{b^2}{\varepsilon_2}K_j^{\mathrm{T}}K_j\Bigg]x \\ &\leq -\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\alpha_j x^{\mathrm{T}}(Q + K_j^{\mathrm{T}}RK_j)x. \end{aligned} \tag{12}$$

Since $Q > 0, R > 0$, the bottom of (12) is negative for all nonzero x, this implies that the system (3) is asymptotically stable, which implies $\lim_{t\to\infty} V(x) = 0$. Moreover, from (12), we have

$$\dot{V}(x) + \sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\alpha_j x^{\mathrm{T}}(Q + K_j^{\mathrm{T}}RK_j)x \leq 0,$$

i.e.,

$$\dot{V}(x) + \sum_{j=1}^{N}\alpha_j x^{\mathrm{T}}(Q + K_j^{\mathrm{T}}RK_j)x \leq 0. \tag{13}$$

We know after some computation that

$$u^{\mathrm{T}}Ru = \sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\alpha_j x^{\mathrm{T}}K_i^{\mathrm{T}}RK_j x \leq \sum_{j=1}^{N}\alpha_j x^{\mathrm{T}}K_j^{\mathrm{T}}RK_j x. \tag{14}$$

Thus from (13) and (14), we have

$$\dot{V}(x) + x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru \leq \dot{V}(x) + \sum_{j=1}^{N}\alpha_j x^{\mathrm{T}}(Q + K_j^{\mathrm{T}}RK_j)x \leq 0,$$

which implies

$$\dot{V}(x) \leq -(x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru). \tag{15}$$

Integrating (15), we obtain

$$\int_0^{+\infty}\dot{V}(x(t))\mathrm{d}t \leq -\int_0^{+\infty} l[x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)]\mathrm{d}t.$$

Therefore

$$J = \int_0^{+\infty}[x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)]\mathrm{d}t \leq V(x(0)) = x^{\mathrm{T}}(0)Px(0).$$

This completes the proof. □

Remark 5. One can see from (8) that the proposed control law has similar structure as fuzzy control. A partition of unity is analogous to the membership function in fuzzy control. What makes a difference here is the modeling methods of nonlinear systems. The fuzzy If–Then rules are empirical and somehow fixed, while we have more freedom in choosing the sample data and constructing a partition of unity. In other words, we can design a more flexible control law.

Remark 6. (9) can be solved using MATLAB (Boyd, Ghaoui, Feron, & Balakrishman, 1994). Let $W = P^{-1}, M_j = K_jW$. Pre- and post-multiplying (9) by W yields

$$\begin{aligned} &WA_i^{\mathrm{T}} + A_iW + B_iM_j + M_j^{\mathrm{T}}B_i^{\mathrm{T}} + (\varepsilon_1+\varepsilon_2)I + \frac{a^2}{\varepsilon_1}W^2 \\ &\quad + \frac{b^2}{\varepsilon_2}M_j^{\mathrm{T}}M_j + WQW + M_j^{\mathrm{T}}RM_j < 0, \end{aligned} \tag{16}$$

while (16) is equivalent to the following LMIs

$$\begin{bmatrix} \Xi_{ij} & aW & bM_j^{\mathrm{T}} & W & M_j^{\mathrm{T}} \\ aW & -\varepsilon_1 I & 0 & 0 & 0 \\ bM_j & 0 & -\varepsilon_2 I & 0 & 0 \\ W & 0 & 0 & -Q^{-1} & 0 \\ M_j & 0 & 0 & 0 & -R^{-1} \end{bmatrix} < 0, \tag{17}$$

where $\Xi_{ij} = WA_i^{\mathrm{T}} + A_iW + B_iM_j + M_j^{\mathrm{T}}B_i^{\mathrm{T}} + (\varepsilon_1+\varepsilon_2)I$.

We can compute W, M_j by solving the above LMIs when they are feasible, then we obtain $P = W^{-1},\ K_j = M_jW^{-1}$.

Theorem 2. *Consider system (3) with the cost function (2). If the following optimization problem*

$$\begin{aligned} &\min_{W>0, M_j, \varepsilon_1>0, \varepsilon_2>0, \eta>0} \quad \eta \\ &\text{subject to} \\ &(17) \quad \text{and} \quad \begin{bmatrix} -\eta & x^{\mathrm{T}}(0) \\ x(0) & -W \end{bmatrix} \leq 0, \end{aligned} \tag{18}$$

has a solution $(W, M_j, \varepsilon_1, \varepsilon_2, \eta)$, then the corresponding law (8) is a suboptimal GCC law from the point of view that the upper bound of the cost function (2) is minimal by adopting (8).

4. Choosing open covering and partition of unity

Constructing a partition of unity is crucial to our main results. If the open covering $\{U_k\}$ of a compact set $\overline{U}$ is composed of some open rectangular domains $U_k = \prod_{i=1}^{n}(a_i^k, b_i^k), k = 1, 2, \ldots, N$, then the partition of unity $\{\alpha_k\}$ subordinate to $\{U_k\}$ can be chosen following the procedure below.

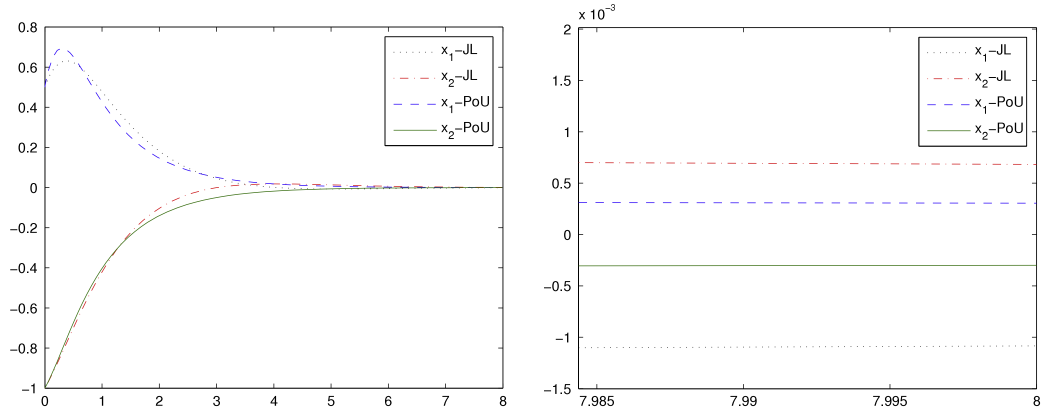
Consider the function $\varphi(\sigma) = \begin{cases} e^{-\frac{1}{\sigma^2}}, & \sigma \neq 0, \\ 0, & \sigma = 0. \end{cases}$ We can prove by direct computation that all derivatives of $\varphi(\sigma)$ exist and are zero at $\sigma = 0$. Hence, $\varphi(\sigma)$ is C^{∞}. Let a_i and b_i denote real numbers such that $a_i < b_i, i = 1, 2, \ldots, N$. Consider the following function

$$g_i(x_i) = \begin{cases} e^{-\left(\frac{1}{x_i-a_i}+\frac{1}{b_i-x_i}\right)^2}, & x_i \in (a_i, b_i), \\ 0, & x_i \notin (a_i, b_i). \end{cases}$$

Recalling the definition of $\varphi(\cdot)$, we can prove that $g_i(x_i)$ is also C^{∞} in x_i, and $g_i(x_i) > 0\ \forall x_i \in (a_i, b_i), g_i(x_i) = 0\ \forall x_i \notin (a_i, b_i)$. Let $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and $g(x) = \prod_{i=1}^{n} g_i(x_i)$, then $g(x) > 0\ \forall x \in \prod_{i=1}^{n}(a_i, b_i)$, $g(x) = 0\ \forall x \notin \prod_{i=1}^{n}(a_i, b_i)$. This shows that if the open covering $\{U_k\}$ of a compact set $\overline{U}$ has the form $\{U_k\} = \prod_{i=1}^{n}(a_i^k, b_i^k), k = 1, 2, \ldots, N$, then we see that the partition of unity subordinate to this open covering can be chosen as the collection of C^{∞} functions $\alpha_k(x) = \frac{g^k(x)}{\sum_{k=1}^{N} g^k(x)}$, where $g^k(x) = \prod_{i=1}^{n} g_i^k(x_i)$,

$$g_i^k(x_i) = \begin{cases} e^{-\left(\frac{1}{x_i-a_i^k}+\frac{1}{b_i^k-x_i}\right)^2}, & x_i \in (a_i^k, b_i^k), \\ 0, & x_i \notin (a_i^k, b_i^k). \end{cases}$$

If the open covering $\{U_k\}$ is composed of some circular domains, say, $U_k = S_k =: S_{r^k}(x^k), k = 1, 2, \ldots, N$, where r^k, x^k are the



(a) Overall state response with $Q = I$.

(b) Locally enlarged version of Fig. 1(a).

Fig. 1. State response using JL and PoU methods, respectively.

radius and center of the circular S_k, respectively, the partition of unity $\{\alpha_k\}$ subordinate to U_k can be chosen following the procedure below.

We can prove similarly that the function $\psi(\sigma) = \begin{cases} e^{-1/\sigma}, & \sigma > 0, \\ 0, & \sigma \le 0 \end{cases}$ is C^∞. Let $\tilde{g}^k(x) = \frac{\psi(r^k - \|x\|)}{\psi(r^k - \|x\|) + \psi(\|x\| - 0.5r^k)}$. Since $\psi(\sigma)$ is C^∞ and the denominator is nonzero, $\tilde{g}^k(x)$ is also a C^∞ function. It is easily seen that $\tilde{g}^k(x) = 0,\ \forall \|x\| \ge r^k$, and $\tilde{g}^k(x) > 0,\ \forall \|x\| < r^k$. Therefore the function $g^k(x) = \tilde{g}^k(x - x^k)$ has the property $g^k(x) > 0,\ \forall x \in S_k$, and $g^k(x) = 0,\ \forall x \notin S_k$.

Choose the collection of C^∞ functions α_k as follows $\alpha_k(x) = \frac{\tilde{g}^k(x - x^k)}{\sum_{j=1}^N \tilde{g}^j(x - x^k)}$. It is easy to verify that $\{\alpha_k, k = 1, 2, \ldots, N\}$ is a partition of unity subordinate to $\{U_k, k = 1, 2, \ldots, N\}$.

For more details, one may refer to Wang et al. (2004).

5. Numerical example

To illustrate the proposed approach, we study in this section an example, which satisfies the aforementioned assumptions.

Example 1. Consider the following nonlinear system. This model with some slight modification (eg. adding some time-delay terms) has been studied in the literature Chen and Liu (2005) and Tanaka, Ikeda, and Wang (1996).

$$\dot{x}_1 = -0.1125x_1 - 0.02x_2 - 0.67x_2^3 + u,$$
$$\dot{x}_2 = x_1,$$

where $x_1 \in [-1.5 \quad 1.5]$, $x_2 \in [-1.5 \quad 1.5]$.

Let $x = [x_1 \quad x_2]^{\mathrm{T}}$. Set $Q = I = \mathrm{diag}\{1, 1\}$, $R = 1$. Let the initial condition be $x(0) = [0.5 \quad -1]^{\mathrm{T}}$. We study this model using three methods.

(I) Using the Jacobian linearization (JL) method.

We linearized the system at equilibrium $[0 \quad 0]^{\mathrm{T}}$, getting the coefficient matrix $A = \begin{bmatrix} -0.1125 & -0.02 \\ 1 & 0 \end{bmatrix}$, the corresponding gain of the LQR $K = \begin{bmatrix} 1.6118 & 0.9802 \end{bmatrix}$. The control law is $u = -Kx$.

(II) Using the proposed partition of unity (PoU) method.

We get the matrices in (3)

$$F(x) = \begin{bmatrix} -0.1125 & -0.02 - 0.67x_2^2 \\ 1 & 0 \end{bmatrix}, \qquad G(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Choose the open coverings as $U_1 = (-1.51 \quad 1.51) \times (-1.51 \quad -1.1)$, $U_2 = (-1.51 \quad 1.51) \times (-1.11 \quad 1.11)$, $U_3 = (-1.51 \quad 1.51) \times (1.1 \quad 1.51)$. The corresponding sample data are chosen as $(0.5, -1.3153)$, $(-0.2, 0.7746)$, $(1, 1.33)$, respectively.

Directly calculating the coefficient matrices in (6), we have

$$A_1 = \begin{bmatrix} -0.1125 & -1.1791 \\ 1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -0.1125 & -0.4420 \\ 1 & 0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -0.1125 & -1.2025 \\ 1 & 0 \end{bmatrix}, \qquad B_1 = B_2 = B_3 = \begin{bmatrix} 1 & 0 \end{bmatrix}^{\mathrm{T}}.$$

Then we have

$$\|(e_{ij}(x))_{n\times n}\| = \left\| F(x) - \sum_{k=1}^{3} \alpha_k A_k \right\| \le \max_{1\le k\le 3} \{\|F_k(x) - A_k\|\} = 0.4087,$$

$$\|(\tilde{e}_{il}(x))_{n\times r}\| = \|G(x) - \sum_{k=1}^{3} \alpha_k B_k\| = 0,$$

where $F_k(x)$ denotes $F(x)$ on U_k, respectively. Thus we obtain $a = 0.4087$, $b = 0$. Because $b = 0$, the corresponding term $\frac{b^2}{\varepsilon_2} K_j^{\mathrm{T}} K_j$ in (9) can be omitted, as a result of which (17) reduces to a 4×4 LMI.

Solving the optimization problem (18), we obtain the cost bound η together with other variables as follows.

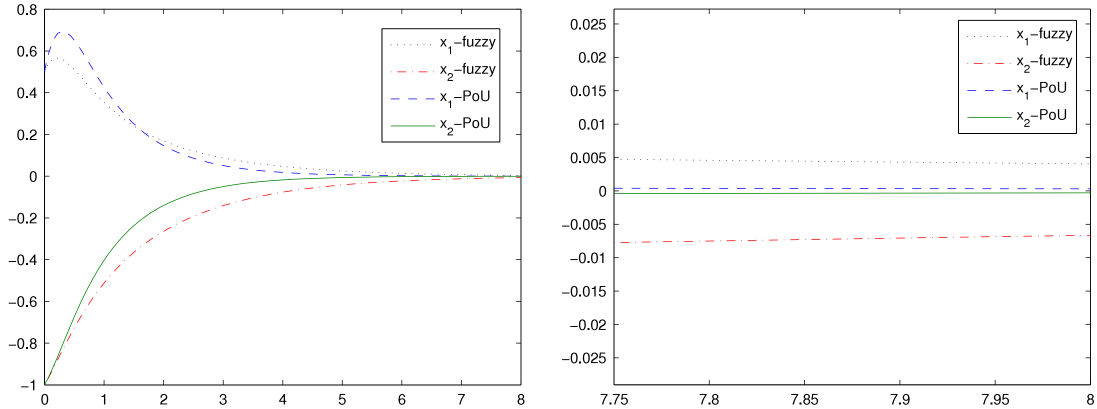
$$P = \begin{bmatrix} 0.4013 & -0.1629 \\ -0.1629 & 0.2149 \end{bmatrix}, \qquad K_1 = \begin{bmatrix} -3.5989 & -2.7279 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} -3.5990 & -2.7280 \end{bmatrix}, \qquad K_3 = \begin{bmatrix} -3.5990 & -2.7281 \end{bmatrix},$$
$$\varepsilon_1 = 0.1359, \qquad \eta = 4.8924.$$

(III) Using the fuzzy control method.

Example 1 with some additional time-delay terms was studied in Chen and Liu (2005). Here we delete those time-delay terms, other parameters remaining the same. We take the GCC law as what given in Chen and Liu (2005).

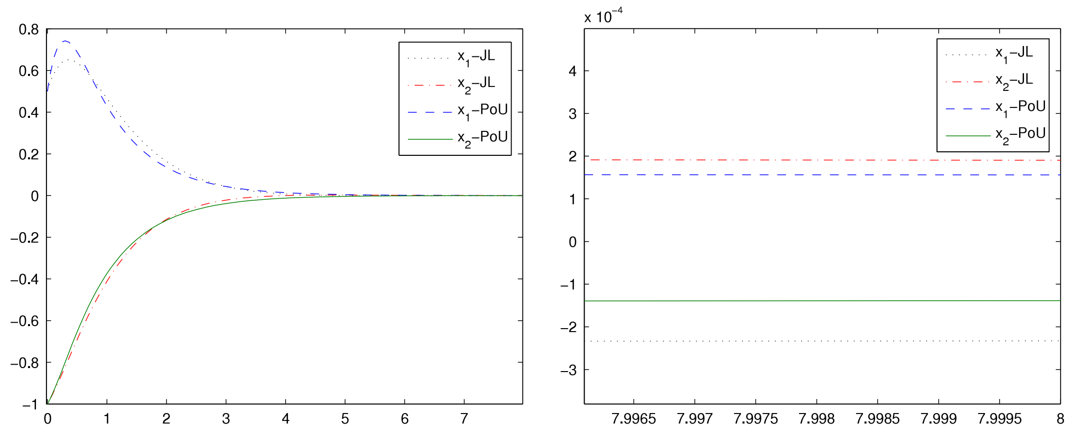
The corresponding comparison simulation results obtained when using methods (I) and (II), as well as when using methods (II) and (III), are shown in Figs. 1 and 2, respectively. When Q is set to $2I$ and $0.5I$, respectively, other parameters remaining the same, additional comparison simulation results obtained when using methods (I) and (II) are provided in Figs. 3 and 4, respectively.

Note that the JL method approximates the nonlinear system only at the equilibrium point and the fuzzy control method ignores the modeling error, while PoU method yields an equivalent model with modeling error, which may better represent the characteristics of the original nonlinear system. We see from the simulation results that in different cases, our method gives faster convergence rate.



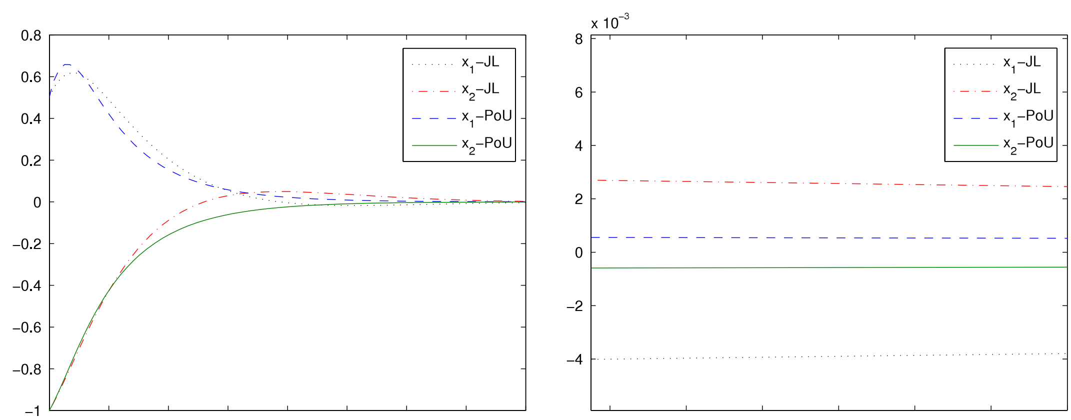
(a) Overall state response with $Q = I$. (b) Locally enlarged version of Fig. 2(a).

Fig. 2. State response using PoU and fuzzy methods, respectively.



(a) Overall state response with $Q = 2I$. (b) Locally enlarged version of Fig. 3(a).

Fig. 3. State response with $Q = 2I$ using JL and PoU methods, respectively.



(a) Overall state response with $Q = 0.5I$. (b) Locally enlarged version of Fig. 4(a).

Fig. 4. State response with $Q = 0.5I$ using JL and PoU methods, respectively.

6. Conclusion

The continuous-time affine nonlinear system is first represented in a linear-like structure with state-dependent coefficient matrices, and later converted into a linear-like system with modeling error using the partition of unity method. The GCC problem is then studied based on the equivalent linear-like system in the presence of modeling error. A sufficient condition on the existence of a GCC law has been proposed using Lyapunov stability theory. A suboptimal GCC law can be obtained after solving a convex optimization problem in terms of LMIs when these LMIs are feasible.

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Donfang Han received his Ph.D. degree in Mathematics from Shantou University, Shantou, China, in 2008. He worked as a Research Associate at the Department of Electronic and Computer Engineering at the Hong Kong University of Science and Technology, from January 2010 to December 2011. He is currently a Lecturer at the School of Mathematics and Statistics, South-Central University for Nationalities, Wuhan, China. His research interests include nonlinear systems, time-delayed systems and networked control systems.

Ling Shi received his B.S. degree in Electrical and Electronic Engineering from the Hong Kong University of Science and Technology in 2002 and Ph.D. degree in Control and Dynamical Systems from California Institute of Technology in 2008. He is currently an Assistant Professor at the Department of Electronic and Computer Engineering at the Hong Kong University of Science and Technology. His research interests include networked control systems, wireless sensor networks and distributed control.