and W_j, $j = 1, 2, 3$, are defined in (9). The last inequality follows $W_j \geq 0$. This concludes the proof.

VI. Conclusion

We take an operator-theoretic point of view to stability analysis of discrete-time linear systems with varying time delays, and propose to tackle the problem via integral quadratic constraint (IQC) analysis. Under this framework, the system is viewed as feedback interconnection of an LTI operator and the so-called "delay difference" operator, for which we derive several novel IQCs based on the assumption that the time-varying delay parameter and its variation are bounded. A set of new stability criteria emerges as the result and their effectiveness are examined via numerical experiments. The results indicate that the approach we propose is in many cases outperforms the state-of-the-art criteria in the literature.

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Convergence and Mean Square Stability of Suboptimal Estimator for Systems With Measurement Packet Dropping

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***Abstract*—We consider remote state estimation over a packet-dropping network. A new suboptimal filter is derived by minimizing the mean squared estimation error. The estimator is designed by solving one deterministic Riccati equation. Convergence of the estimation error covariance and mean square stability of the estimator are proved under standard assumptions. It is shown that the new estimator has smaller error covariance and has wider applications when compared with the linear minimum mean squared error estimator. One of the key techniques adopted in this technical note is the introduction of the innovation sequence for the multiplicative noise systems.**

***Index Terms*—Convergence, discrete-time system, mean square stability, packet dropping, Riccati difference equation, suboptimal estimation.**

I. Introduction

The problem of state estimation is of great importance in various applications ranging from tracking, detection and control [1]. For many linear stochastic models, a useful tool is the standard Kalman filtering theory which has a wide spectrum of applications.

The Kalman filter is well studied in control theory when there is no information loss [2]. However, the recent trend of utilizing networks for transmitting measurement data introduces some interesting new problems due to the unreliable characteristics of networks such as random data packet delays and drops. How does packet dropping affect the performance of an estimator is of significant interest.

The early work [3] considered the linear minimum mean squared error (LMMSE) estimation. By modeling the uncertainty as a sequence of i.i.d. binary random variables indicating the signal availability, the author derived a recursion similar to the Kalman filter utilizing the statistics of the unobserved binary uncertainty sequence. In [4] the authors gave conditions for obtaining recursive filtering when the uncertainty sequence is not necessarily i.i.d. Asymptotic stability of the LMMSE filter was established in [5] when the packet arrival sequence is i.i.d. with known arrival probability. Since in this case the estimation covariance is governed by a deterministic equation, one can perform stability analysis by constructing an equivalent linear system without packet dropping. However, LMMSE filter in the aforementioned literatures

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[3]–[6] only makes use of the statistics of the unobserved binary uncertainty sequence, and the solution to the estimator involves both a Riccati equation and a Lyapunov equation. Note that solving the additional Lyapunov equation introduces extra cost of computation, and moreover, restricts the application to stable systems only when refer to infinite horizon case [5].

On the other hand, Kalman filter with intermittent observations by using time-stamped data packet has attracted much attention in recently years [7]–[10] (we shall call it as *intermittent Kalman filter* in this technical note). The authors in [7] has shown that, for an unstable system with i.i.d. Bernoulli packet drops, there exists a critical threshold such that the expected value of the error covariance will be bounded if the packet arrival rate exceeds this threshold, but will diverge otherwise. This approach was extended by [8] to multi-sensor scenario. In these results, the packet arrival sequence is known at the estimator and this leads to a random Riccati equation involving the packet arrival indicator sequence, which differs from the deterministic recursion for the covariance function in [3]–[6]. The stability analysis of Kalman filter with binary Markovian packet dropping can be found in [9] where the notion of peak covariance stability was first introduced. It should be pointed out that both the error covariance matrix iteration and Kalman filter updates are stochastic and depend on the random arrivals of the measurements. It has been proved difficult to analyze the convergence of the stochastic Riccati equation and the mean square stability of the estimator. The existing results are limited to boundary analysis.

Motivated by this, we propose a suboptimal estimator under a new performance index. The proposed estimator can be seen as a tradeoff in performance between the LMMSE filter and the intermittent Kalman filter. The new estimator shall improve the performance of LMMSE Kalman filter and possess better properties of convergence and stability than the intermittent Kalman filter.

The remainder of the technical note is organized as follows. Section II provides the problem statement. The suboptimal filter is derived in Section III, and the convergence and stability of the filter are proved under standard assumptions. Differences between our estimator and previous estimators are discussed. In Section IV, an example is given to demonstrate the effectiveness of the approach. Section V concludes the technical note. Some proofs for the main results are provided in the appendix.

Notation: Throughout this technical note, a real symmetric matrix $P > 0 (\geq 0)$ means that P is a positive definite (or positive semi-definite) matrix, and $A > (\geq) B$ means $A - B > (\geq) 0$. Matrix I denotes an identity matrix of appropriate dimension. The superscript "T" represents the transpose, $\mathcal{R}^n$ denotes the n-dimensional Euclidean space. $E\{\cdot\}$ stands for the mathematical expectation operator. $*$ is used as an ellipsis for terms that are induced by symmetry. Matrices, if the dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

II. Problem Statement

Consider the following system:

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + \mathbf{w}(k) \tag{1}$$

$$\mathbf{y}(k) = \gamma(k)H\mathbf{x}(k) + \mathbf{v}(k) \tag{2}$$

where $\mathbf{x}(k) \in R^n$ and $\mathbf{y}(k) \in R^m$ are respectively the system state and measurement, $\mathbf{w}(k)$ and $\mathbf{v}(k)$ are respectively the system noise and measurement noise with zero mean and covariances $E\{\mathbf{w}(k)\mathbf{w}^T(j)\} = Q\delta_{k,j}$, $E\{\mathbf{v}(k)\mathbf{v}^T(j)\} = R\delta_{k,j}$, where $\delta_{k,j}$ is the Kronecker Delta function. The initial state $\mathbf{x}(0)$ is a random vector with mean μ_0 and covariance $E\{[\mathbf{x}(0) - \mu_0][\mathbf{x}(0) - \mu_0]^T\} = P_0$. The following assumptions are made on the packet arrival indicator $\gamma(k)$ throughout the technical note.

Assumption 1: $\gamma(k)$ is a scalar quantity taking on values of 0 and 1 with $Pr\{\gamma(k) = 1\} = q$, $Pr\{\gamma(k) = 0\} = 1 - q$, and the random processes $\mathbf{w}(k), \mathbf{v}(k), \gamma(k)$ for all k and the initial state $\mathbf{x}(0)$ are mutually independent.

Assumption 2: $\gamma(k)$ is assumed to be observed at every time instant k by employing the time-stamp technique. That is $\gamma(k)$ together with observation $\mathbf{y}(k)$ are available in the estimator design.

Before describing the problems, we introduce the sequence $\mathbf{e}(k)$ associated with the measurement $\mathbf{y}(k)$

$$\mathbf{e}(k) \triangleq \mathbf{y}(k) - \gamma(k)H\hat{\mathbf{x}}(k|k-1) \tag{3}$$

where

$$\hat{\mathbf{x}}(k+1|k) = A\hat{\mathbf{x}}(k|k-1) + K_p(k)\mathbf{e}(k), \ \hat{\mathbf{x}}(0|-1) = \mu_0 \tag{4}$$

and $K_p(k)$ is chosen such that the following is minimized:

$$E\left\{[\mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1|k)]\,[\mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1|k)]^T\right\} \tag{5}$$

where the expectation is taken over $\mathbf{w}$, $\mathbf{v}$ and γ.

Recall [10] we have the following results:

Lemma 1: The sequence $\mathbf{e}(k)$ defined in (3) is mutually uncorrelated noise with zero mean.

Remark 1: It is clear that the sequence $\mathbf{e}(k)$ defined in (3) is different from the standard Kalman innovation for additive-noise system [11] but they posses the similarity of being mutually uncorrelated and having zero mean. Therefore we shall refer $\mathbf{e}(k)$ as *innovation* in this technical note for consistency.

We now formulate the problem with the innovation sequence $\{\mathbf{e}(\cdot)\}$.

Suboptimal Filter: Find an estimator $\hat{\mathbf{x}}(k+1|k)$ of $\mathbf{x}(k+1)$ in the form of (4), where the gain $K_p(k)$ is chosen such that (5) is minimized.

Remark 2: The above filtering problem is different from the intermittent Kalman filter studied in [7] where the expectation is only taken over on $\mathbf{w}$ and $\mathbf{v}$, and γ is assumed to be known.

Remark 3: The above estimation problem is also different from the LMMSE Kalman filtering problem. In fact, the measurement (2) can be rewritten as

$$\mathbf{y}(k) = qH\mathbf{x}(k) + [\gamma(k) - q]\,H\mathbf{x}(k) + \mathbf{v}(k). \tag{6}$$

Then the LMMSE estimation problem is to find the estimator $\hat{\mathbf{x}}(k+1|k)$ in the form of

$$\hat{\mathbf{x}}(k+1|k) = A\hat{\mathbf{x}}(k|k-1) + \tilde{K}_p(k)\,[\mathbf{y}(k) - qH\hat{\mathbf{x}}(k|k-1)] \tag{7}$$

and $\tilde{K}_p(k)$ is chosen such that (5) is minimized.

III. Solution to the Suboptimal Estimator

A. Calculation of the Suboptimal Estimator

The following result gives the suboptimal gain matrix $K_p(k)$ of (4).

Theorem 1: For the given systems (1), (2), the filter defined in (4) is given by

$$\hat{\mathbf{x}}(k+1|k) = [A - \gamma(k)K_p(k)H]\hat{\mathbf{x}}(k|k-1) + K_p(k)\mathbf{y}(k),$$
$$\hat{\mathbf{x}}(0|-1) = \mu_0 \tag{8}$$

where the estimator gain $K_p(k)$ is calculated as

$$K_p(k) = qAP(k)H^TM^{-1}(k) \tag{9}$$

$$M(k) = qHP(k)H^T + R \tag{10}$$

and $P(k)$, the covariance of estimation error, is the solution to the generalized Riccati equation

$$P(k+1) = AP(k)A^T + Q - K_p(k)M(k)K_p^T(k),\quad P(0) = P_0. \tag{11}$$

Proof: Let $\tilde{\mathbf{x}}(k|k-1) = \mathbf{x}(k) - \hat{\mathbf{x}}(k|k-1)$. From (4), we have

$$\tilde{\mathbf{x}}(k+1|k) = A\tilde{\mathbf{x}}(k|k-1) - K_p(k)\gamma(k)H\tilde{\mathbf{x}}(k|k-1) + \mathbf{w}(k) - K_p(k)\mathbf{v}(k). \tag{12}$$

Then it follows from (12) that:

$$\begin{aligned}
&\tilde{\mathbf{x}}(k+1|k)\tilde{\mathbf{x}}^T(k+1|k)\\
&= \{A - K_p(k)\gamma(k)H\}\,\tilde{\mathbf{x}}(k|k-1)\\
&\quad\times \tilde{\mathbf{x}}^T(k|k-1)\,\{A - K_p(k)\gamma(k)H\}^T\\
&\quad- \{A - K_p(k)\gamma(k)H\}\,\tilde{\mathbf{x}}(k|k-1)\mathbf{v}^T(k)K_p^T(k)\\
&\quad+ \{A - K_p(k)\gamma(k)H\}\,\tilde{\mathbf{x}}(k|k-1)\mathbf{w}^T(k)\\
&\quad- K_p(k)\mathbf{v}(k)\tilde{\mathbf{x}}^T(k|k-1)\,\{A - K_p(k)\gamma(k)H\}^T\\
&\quad+ K_p(k)\mathbf{v}(k)\mathbf{v}^T(k)K_p^T(k) - K_p(k)\mathbf{v}(k)\mathbf{w}^T(k)\\
&\quad- \mathbf{w}(k)\mathbf{v}^T(k)K_p^T(k) + \mathbf{w}(k)\mathbf{w}^T(k)\\
&\quad+ \mathbf{w}(k)\tilde{\mathbf{x}}^T(k|k-1)\,\{A - K_p(k)\gamma(k)H\}^T.
\end{aligned} \tag{13}$$

Taking expectation with respect to $\gamma(k)$, $\mathbf{w}(k)$ and $\mathbf{v}(k)$ on both sides of (13) yields

$$\begin{aligned}
&E\left\{[\tilde{\mathbf{x}}(k+1|k)]\,[\tilde{\mathbf{x}}(k+1|k)]^T\right\}\\
&= AP(k)A^T + Q - q^2AP(k)H^TM^{-1}(k)\left[AP(k)H^T\right]^T\\
&\quad+ \left[K_p(k) - K_p^*(k)\right]M(k)\left[K_p(k) - K_p^*(k)\right]^T
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
P(k) &= E\left\{\tilde{\mathbf{x}}(k|k-1)\tilde{\mathbf{x}}^T(k|k-1)\right\},\\
K_p^*(k) &= qAP(k)H^TM^{-1}(k),\\
M(k) &= qHP(k)H^T + R.
\end{aligned}$$

It is obvious that $E\{\tilde{\mathbf{x}}(k+1|k)\tilde{\mathbf{x}}^T(k+1|k)\}$ will be minimized precisely if we choose $K_p(k) = K_p^*(k)$. Therefore the proof is completed. ■

B. Convergence and Stability of the Estimator

In this subsection, we study the convergence and stability of the proposed estimator. Similar to our previous work [12], we assume the following in this technical note.

Assumption 3: $(A^T, qH^T, 0, H^T)$ $(0 < q < 1)$ is stabilizable.

Assumption 4: $(A^T, 0, Q^{1/2})$ is exactly observable.

Remark 4: Recall from [12] that, (A^T, H^T, A_0^T, H_0^T) is called stabilizable in the mean square sense if there exists a feedback control $u(k) = Kx(k)$ with K being a constant matrix, such that for any $x_0 \in R^n$, the closed-loop system

$$x(k+1) = [A^T + H^TK]x(k) + \left[A_0^T + H_0^TK\right]x(k)\omega(k),\quad x(0) = x_0 \tag{15}$$

is asymptotically mean square stable. Here $\omega(k)$ in (15) is a wide sense stationary, second-order process with $E\{\omega(k)\} = 0$ and $E\{\omega(k)\omega(j)\} = \sigma\delta_{kj}$.

Consider the stochastic system

$$x(k+1) = A^Tx(k) + A_0^Tx(k)\omega(k) \tag{16}$$

$$y(k) = C^Tx(k) \tag{17}$$

(A^T, A_0^T, C^T) is called exactly observable if

$$y(k) \equiv 0,\quad a.s.\ \forall\, k \in \{0, 1, \cdots\} \Longrightarrow x_0 = 0.$$

Theorem 2: Under the Assumption 3–4, with an arbitrary but fixed initial nonnegative symmetric $P(0)$, the matrix $P(k)$ abided by (11) converges to a unique positive definite matrix P. In other words, we have $lim_{k\to\infty}P(k) = P > 0$, $lim_{k\to\infty}K_p(k) = K_p \equiv qAPH^T[qHPH^T + R]^{-1}$.

Proof: See Appendix A. ■

Following Theorem 2, when $k \to \infty$, the suboptimal filter of (8) becomes

$$\hat{\mathbf{x}}(k+1|k) = [A - \gamma(k)K_pH]\,\hat{\mathbf{x}}(k|k-1) + K_p\mathbf{y}(k). \tag{18}$$

Now we present the results on mean square stability for the filter (18).

Theorem 3: Under the Assumption 3–4, the suboptimal estimator (18) is mean square stable.

Proof: Note that (18) can be further rewritten as

$$\hat{\mathbf{x}}(k+1|k) = [A - qK_pH]\hat{\mathbf{x}}(k|k-1) + K_p\mathbf{y}(k) + \bar{\gamma}(k)[-K_pH]\hat{\mathbf{x}}(k|k-1) \tag{19}$$

where $\bar{\gamma}(k) = \gamma(k) - q$ is a random scalar variable with zero mean and covariance $q(1-q)$. In the proof of Theorem 2 it has been shown that

$$[A - qK_pH]P[A - qK_pH]^T + q(1-q)K_pHPH^TK_p^T - P < 0. \tag{20}$$

Thus we conclude that the system

$$\hat{\mathbf{x}}(k+1|k) = [A - qK_pH]\hat{\mathbf{x}}(k|k-1) + \bar{\gamma}(k)[-K_pH]\hat{\mathbf{x}}(k|k-1) \tag{21}$$

is mean square stable [13]. Hence (18) is a mean square stable estimator. ■

Corollary 1: If $(A, Q^{1/2})$ is reachable and (A, H) is detectable. Then for any $q \geq q_0$, the matrix $P(k)$ of (11) converges to a constant matrix P, where q_0 is given by the solution to the following optimization problem:

$$q_0 = argmin_q \Phi_q(W, \bar{X}) > 0,\ 0 < W \leq I,\ 0 < q \leq 1 \tag{22}$$

$$\Phi_q(W, \bar{X}) = \begin{bmatrix} W & * & * & * & * \\ A^TW - qH^T\bar{X}^T & W & * & * & * \\ \sqrt{(1-q)q}H^T\bar{X}^T & 0 & W & * & * \\ R^{\frac{1}{2}}\bar{X}^T & 0 & 0 & I & * \\ FW & 0 & 0 & 0 & I \end{bmatrix} \tag{23}$$

where F is a real matrix satisfying $Q = F^TF$. Furthermore, the estimator (18) is mean square stable.

Proof: See Appendix B. ■

Remark 5: It is readily known that $(A, Q^{1/2})$ is reachable is equivalent to Assumption 4.

C. Discussions on the Proposed Estimator and Previous Estimators

The study on Kalman filtering problem for packet dropping can be tracked back in the 60's. The earlier works focus on the LMMSE (linear minimum mean squared error) estimation [3]–[5] and the recent works on the standard Kalman filter using time-stamp technique [7]. The presented filter (8) is derived by combining the LMMSE approach and time-stamp technique. In this subsection, we compare the proposed estimator with the previous ones.

1) Comparison with LMMSE Kalman Filter: From Remark 3, the LMMSE Kalman filtering gain $\tilde{K}_p(k)$ is calculated as [14] $\tilde{K}_p(k) = qAS(k)H^T[q^2HS(k)H^T+q(1-q)HD(k)H^T+R]^{-1}$, where $D(k)$ and $S(k)$ are given by

$$D(k+1) = AD(k)A^T + Q \tag{24}$$

$$\begin{aligned} S(k+1) = {} & AS(k)A^T + Q - q^2AS(k)H^T \\ & \times \left[q^2HS(k)H^T + q(1-q)HD(k)H^T + R\right]^{-1} \\ & \times HS(k)A^T \end{aligned} \tag{25}$$

where $S(k)$ is the error covariance of the LMMSE estimator. Thus the design of the LMMSE estimator (7) involves both the solution to the Riccati difference (25) and the solution to a Lyapunov (24). Due to the fact that the Lyapunov equation is solvable only in the case when the system is stable, the LMMSE estimator for systems (1), (2) is only applicable to stable systems. Note that the solvable condition of the presented estimator has been reduced to a weaker condition of Assumption 1 and 2.

Moreover, the proposed estimator has smaller estimation error than the LMMSE estimator since the new estimator exploits additional information on the arrival sequence.

Lemma 2: With the same initial condition $D(0) = S(0) = P(0)$, one has

$$P(k+1) \leq S(k+1) \leq D(k+1) \tag{26}$$

where $P(k+1)$, $S(k+1)$ and $D(k+1)$ are as in (11), (24) and (25), respectively.

Proof: We complete the proof by induction. Note that $S(0) = D(0)$, it follows readily from (24), (25) that $S(1) \leq D(1)$. Assume $S(k) \leq D(k)$, then

$$S(k+1) \leq AS(k)A^T + Q \leq AD(k)A^T + Q = D(k+1).$$

Next we prove $P(k+1) \leq S(k+1)$. Since $S(0) = P(0)$, we have $P(1) \leq S(1)$. Now suppose $P(k) \leq S(k)$, then

$$\begin{aligned} & P(k+1) \\ & = [A-K_p(k)qH]\,P(k)\,[A-K_p(k)qH]^T+Q \\ & \quad +q(1-q)K_p(k)HP(k)H^TK_p^T(k)+K_p(k)RK_p^T(k) \\ & \leq \left[A-\bar{K}_p(k)qH\right]P(k)\left[A-\bar{K}_p(k)qH\right]^T+Q \\ & \quad +q(1-q)\bar{K}_p(k)HP(k)H^T\bar{K}_p^T(k)+\bar{K}_p(k)R\bar{K}_p^T(k) \\ & \leq \left[A-\bar{K}_p(k)qH\right]S(k)\left[A-\bar{K}_p(k)qH\right]^T+Q \\ & \quad +q(1-q)\bar{K}_p(k)HS(k)H^T\bar{K}_p^T(k)+\bar{K}_p(k)R\bar{K}_p^T(k) \\ & \leq AS(k)A^T+Q-q^2AS(k)H^T \\ & \quad \times\left[q^2HS(k)H^T+q(1-q)HD(k)H^T+R\right]^{-1}HS(k)A^T \\ & = S(k+1) \end{aligned} \tag{27}$$

where $K_p(k)$ is given by (9), $\bar{K}_p(k) = qAS(k)H^T \times [qHS(k)H^T + R]^{-1}$. Thus the proof is completed. ■

2) Comparison with Intermittent Kalman Filter: Although the intermittent Kalman filter has received much attention in recent years, the analysis of the estimator is only limited to the boundary analysis for estimation-error covariance due to the complexity of discussions [8], [9]. On the other hand, we have shown in Section III that the proposed estimator in this technical note is mean square stable and the covariance of estimation error converges to a positive definite matrix under standard assumptions. However, we would like to point out that the proposed estimator has larger estimation error covariance than intermittent Kalman filter in time-stamp case. Thus the proposed estimator can be seen as a tradeoff in performance and analysis between the LMMSE filter and the intermittent Kalman filter.

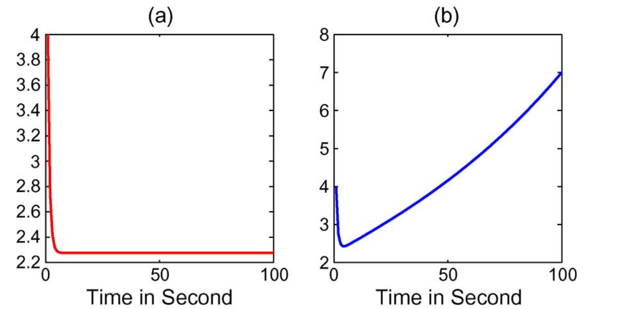

Fig. 1. (a) Covariance of proposed estimator; (b) Covariance of LMMSE estimator.

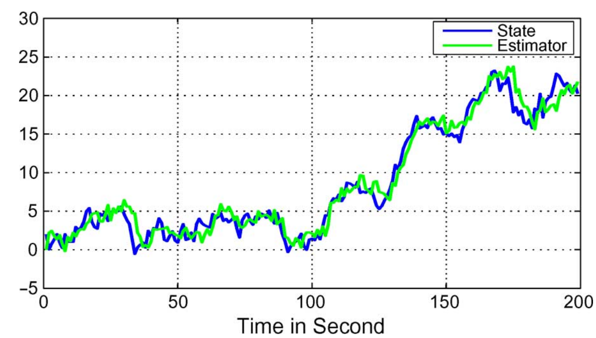

Fig. 2. Tracking performance of the proposed estimator (30).

IV. Numerical Example

Consider the linear discrete-time system

$$\mathbf{x}(k+1) = 1.01\mathbf{x}(k) + \mathbf{w}(k), \mathbf{x}(0) = 2 \tag{28}$$

$$\mathbf{y}(k) = -0.7\gamma(k)\mathbf{x}(k) + \mathbf{v}(k) \tag{29}$$

where $\mathbf{w}(k)$ and $\mathbf{v}(k)$ are Gaussian random noises with zero means and covariances 1, and $\gamma(k)$ is the packet arrival indicator with $Pr\{\gamma(k) = 1\} = 0.9$ and $Pr\{\gamma(k) = 0\} = 0.1$. We design the LMMSE estimator and the proposed new estimator for the above system. The estimation error covariance matrix $P(k)$ in (11) and $S(k)$ in (25) are shown in Fig. 1 (with its tracking performance in Fig. 2), which shows that the covariance of our proposed estimator is asymptotically convergent while the covariance of the LMMSE estimator is divergent. Furthermore, the covariance of estimation error of our proposed estimator converge to 2.2745. As a result, the constant gain K_p is -0.7225 and the estimator is

$$\hat{\mathbf{x}}(k+1|k) = [1.01 - 0.50575\gamma(k)]\,\hat{\mathbf{x}}(k|k-1) - 0.7225\mathbf{y}(k). \tag{30}$$

V. Conclusion

A new approach to suboptimal estimation over a packet-dropping network has been proposed by minimizing the mean squared estimation error where the mean has been taken over the system noise, measurement noise and random packet arrivals. Under this performance index, it has been shown that the derived estimation gain for filtering is constant which allows us to analyze the convergence and stability for the proposed estimator under standard assumptions. We have also shown in theory and simulation that the proposed estimator has wider applications and better performance than the LMMSE filter.

APPENDIX A
PROOF OF THEOREM 2

Proof: In view of Assumption 3, there is a matrix K_e such that $(A - K_e qH, -K_e H)$ is mean square stable. Define a fictitious suboptimal as follows:

$$\hat{\mathbf{x}}^e(k+1|k) = A\hat{\mathbf{x}}^e(k|k-1) + K_e\left\{\mathbf{y}(k) - \gamma(k)H\hat{\mathbf{x}}^e(k|k-1)\right\}. \quad (31)$$

Let $\tilde{\mathbf{x}}^e(k+1|k) = \mathbf{x}(k+1) - \hat{\mathbf{x}}^e(k+1|k)$, and $P^e(k+1) = E[\tilde{\mathbf{x}}^e(k+1|k)\tilde{\mathbf{x}}^{eT}(k+1|k)]$. Combining (1) and (31) we obtain

$$\tilde{\mathbf{x}}^e(k+1|k) = [A - \gamma(k)K_e H]\,\tilde{\mathbf{x}}^e(k|k-1) + \mathbf{w}(k) - K_e\mathbf{v}(k). \quad (32)$$

Thus one has

$$\begin{aligned} P^e(k+1) = {} & [A - K_e qH]P^e(k)[A - K_e qH]^T + K_e R K_e^T \\ & + q(1-q)K_e H P^e(k) H^T K_e^T + Q. \end{aligned} \quad (33)$$

Considering that $(A - K_e qH, -K_e H)$ is mean square stable, we obtain that $P^e(k+1)$ in (33) has a bounded solution for any initial condition. By its suboptimality, $P^e(k+1) \geq P(k+1) \geq 0$ in general, thus the bound on $P(k+1)$ is obtained. On the other hand, let $P^{Q_0}(k)$ and $K_P^{Q_0}(k)$ stand for the covariance matrix gain satisfying (11) and gain matrix abiding by (9), both with an arbitrary initial condition Q_0. Then similar to [15] we can prove that $P^0(k+1) \geq P^0(k)$ for $\forall k$. As shown previously, $P^0(k)$ is bounded and monotonically increasing with k, which implies $lim_{k\to\infty} P^0(k) = P \geq 0$. Now taking limits in (9)–(11) with initial condition 0 one obtains

$$\begin{aligned} K_p &= qAPH^T M^{-1}, M = qHPH^T + R, \\ P &= APA^T + Q - K_p M K_p^T. \end{aligned}$$

Along the same line of the proof in [12], we conclude that $P > 0$ due to the fact that $(A^T, 0, Q^{1/2})$ is exactly observable. Note that it follows from (34) that

$$[A - K_p qH]P[A - K_p qH]^T + q(1-q)K_p H P H^T K_p^T - P < 0. \quad (34)$$

According to [13], we conclude that $(A - K_p qH, -K_p H)$ is mean square stable.

In the following, we show that the Riccati iteration initialized at $P(0) = R_0 \geq P$ also converges, and to the same limit P. To prove this, we consider the following operators:

$$\begin{aligned} g_q(Y) = {} & AYA^T + Q - q^2 AYH^T \\ & \times [qHYH^T + R]^{-1}[AYH^T]^T \end{aligned} \quad (35)$$

$$\begin{aligned} \phi_q(L, Y) = {} & [A - LqH]Y[A - LqH]^T \\ & + q(1-q)LHYH^T L^T + Q + LRL^T. \end{aligned} \quad (36)$$

Assume $Y \in \mathbb{S} = \{S \in \mathbb{R}^{n\times n} | S \geq 0\}$. According to the similar idea in [7], the following facts are true.

i): With $L_Y = qAYH^T[qHYH^T + R]^{-1}$, $g_q(Y) = \phi_q(L_Y, Y)$.

ii): $g_q(Y) = min_L \phi_q(L, Y) \leq \phi_q(L, Y), \forall L$.

iii): If $Y_1 \leq Y_2$, then $g_q(Y_1) \leq g_q(Y_2)$.

By making use of (35) and the fact iii), we have $P^{R_0}(1) = g_q(P^{R_0}(0)) = g_q(R_0) \geq g_q(P) = P$. A simple calculation using the mathematical induction reveals that $P^{R_0}(k) \geq P$ for $\forall k$. Moreover, it follows from (35), (36) and i), ii) and iii) that:

$$\begin{aligned} 0 \leq & P^{R_0}(k+1) - P \\ = & \phi_q\left(K_p^{R_0}(k), P^{R_0}(k)\right) - \phi_q(K_p, P) \\ \leq & \phi_q\left(K_p, P^{R_0}(k)\right) - \phi_q(K_p, P) \\ = & [A - K_p qH]\left[P^{R_0}(k) - P\right][A - K_p qH]^T \\ & + q(1-q)K_p H\left[P^{R_0}(k) - P\right][K_p H]^T. \end{aligned} \quad (37)$$

Since $(A - K_p qH, -K_p H)$ is stable, by taking limit on (37) one has

$$0 \leq \lim_{k\to+\infty} P^{R_0}(k+1) - P = 0. \quad (38)$$

Thus we have proved that the Riccati iteration initialized at $P(0) = R_0$ also converges to the same limit P. Finally, we demonstrate that $P(k)$ in (11) also converges to the same limit P under arbitrary initial value $P(0) = P_0$. Define $R_0 = P_0 + P$. Consider the three Riccati iterations, initialized with 0, P_0 and R_0. Note that $0 \leq P_0$. Assume that $P^0(k+1) \leq P^{P_0}(k+1)$, then $P^0(k+2) = g(P^0(k+1)) \leq g(P^{P_0}(k+1)) = P^{P_0}(k+2)$. Hence $\forall\, k$, $P^0(k+2) \leq P^{P_0}(k+2)$ holds. Similarly we obtain that $P^{P_0}(k+2) \leq P^{R_0}(k+2)$. In other words, $\forall k$, $P^0(k+2) \leq P^{P_0}(k+2) \leq P^{R_0}(k+2)$. Noting that the Riccati equations $P^0(k+2)$ and $P^{R_0}(k+2)$ in (11) converge to the same P, thus

$$\begin{aligned} P = & \lim_{k\to\infty} P^0(k+2) \leq \lim_{k\to\infty} P^{P_0}(k+2) \\ & \leq \lim_{k\to\infty} P^{R_0}(k+2) = P. \end{aligned} \quad (39)$$

And the proof is completed. ■

APPENDIX B
PROOF OF COROLLARY 1

Proof: When $q = 1$, under the conditions that $(A, Q^{1/2})$ is reachable and (A, H) is detectable, it has been proven in [15] that the matrix $P(k)$ of (11) converges to a unique positive definite matrix P which satisfies the algebraic Riccati equation

$$P = APA^T + Q - APH^T[HPH^T + R]^{-1}HPA^T. \quad (40)$$

Set $Y = W^{-1}$, $K = W^{-1}\bar{X}$, it can be deduced from $\Phi_{q_0}(W, \bar{X}) < 0$ after elementary transformation that

$$\begin{bmatrix} Y & * & * \\ YA^T - q_0 YH^T K^T & Y & * \\ \sqrt{(1-q_0)q_0}\,YH^T K^T & 0 & Y \end{bmatrix} > 0 \quad (41)$$

which is also equivalent to that there exists a positive definite matrix Y such that

$$[A - q_0 KH]Y[A - q_0 KH]^T + q_0(1-q_0)KHYH^T K^T - Y < 0. \quad (42)$$

Therefore $(A^T, 0, q_0 H^T, H^T)$ $(0 < q_0 < 1)$ is stabilizable from (42). On the other hand, $(A, Q^{1/2})$ is reachable is equivalent to $(A^T, 0, Q^{1/2})$ is exactly observable. Thus based on Theorem 2, one has that $P(k)$, which satisfies

$$\begin{aligned} P(k+1) = {} & AP(k)A^T + Q - q_0^2 AP(k)H^T \\ & \times [q_0 HP(k)H^T + R]^{-1} HP(k)A^T \end{aligned} \quad (43)$$

converges to a positive definite matrix P, and P satisfies the following algebraic Riccati equation:

$$P = APA^T + Q - q_0^2 APH^T[q_0 HPH^T + R]^{-1}HPA^T. \quad (44)$$

Finally we will find the minimum q_0 such that for any $1 > q \geq q_0$, $P(k)$ of (11) converges. In order to do this, similar to [7], we can prove the following statements are equivalent.

a): $\exists Y > 0$ such that $Y > g_q(Y)$.

b): $\exists L, Y > 0$ such that $Y > \phi_q(L, Y)$.

c): $\exists \bar{X}$ and $0 < W \leq I$ such that

$$\Phi_q(W, \bar{X}) = \begin{bmatrix} W & * & * & * & * \\ A^T W - qH^T \bar{X}^T & W & * & * & * \\ \sqrt{(1-q)q} H^T \bar{X}^T & 0 & W & * & * \\ R^{\frac{1}{2}} \bar{X}^T & 0 & 0 & I & * \\ FW & 0 & 0 & 0 & I \end{bmatrix} > 0.$$

Moreover, we can prove that $g_q(Y)$ is monotone in q, that is, $g_{q_1}(Y) \geq g_{q_2}(Y), \forall 0 < q_1 \leq q_2 < 1, \forall Y > 0$. Combining the facts a), b) and making use of the monotonicity of $g_q(Y)$, we get $(A^T, 0, qH^T, H^T)$ $(q_0 \leq q < 1)$ is also stabilizable under the condition of $(A^T, 0, q_0 H^T, H^T)$ is stabilizable. Also, we can deduce $\Phi_q(W, \bar{X}) > 0$ from $\Phi_{q_0}(W, \bar{X}) > 0$ by combing the facts a), c) and making use of the monotonicity of $g_q(Y)$. Thus we have proved that under the condition $\Phi_q(W, \bar{X}) > 0$ and $(A, Q^{1/2})$ is reachable, $P(k)$ satisfying (11) converges to a positive definite matrix, and the minimum q_0 is given by the solution to the optimization problem (22). As for the mean square stable estimator, one can follow the proof of Theorem 3. Thus the proof is completed. ■

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Power Control in Wireless Networks: Stability and Delay Independence for a General Class of Distributed Algorithms

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***Abstract*—We show that for a general class of distributed power control algorithms in wireless networks, if a feasible steady state power allocation exists, this is asymptotically stable for *arbitrary* gains and time varying heterogeneous delays. The analysis exploits certain contraction properties of the interference in such algorithms, and makes use of Lyapunov Razumikhin functions to address the infinite dimensional character of the problem.**

***Index Terms*—Decentralized control, delays, large scale systems, network analysis and control.**

I. INTRODUCTION

Efficient control of power is an important part of the design of wireless systems. On the one hand, the power transmitted by individual nodes must be high enough to ensure a reliable connection, but at the same time, this causes interference to neighboring nodes and reduces battery lifetime. This is a tradeoff that has triggered research in this area from an early stage e.g. [1]–[6] and more recent control theoretic approaches as in [7]–[9]. Furthermore, the need for power control schemes to be distributed is a key requirement in large scale networks, i.e. power update rules should be based on local interference measurements, rather than centralized control. One of the most well known distributed algorithms was originally proposed in [2], with its asynchronous version analyzed in [10]. This is essentially a linear update scheme that converges to user specific signal-to-interference-ratio requirements. Convergence is guaranteed if a feasible such power allocation exists, but the system will otherwise diverge. Therefore, nonlinearites in the form of power constraints, or other power assignment rules, often need to be incorporated. This has led to the general framework introduced by Yates in [4], which is based on some generic assumptions on the effective interference from other users.

Delays, which can be heterogeneous and time varying, are inevitably present in the communication of interference measurements in such algorithms. These cannot be neglected in realistic models as it is well known that they can potentially lead to instabilities and oscillatory behaviors. Nevertheless, they render continuous time models of such nonlinear control laws infinite dimensional, something that adds a

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