Correspondences

Scheduling Two Gauss–Markov Systems: An Optimal Solution for Remote State Estimation Under Bandwidth Constraint

Ling Shi and Huanshui Zhang

***Abstract*—We consider scheduling two Gauss–Markov systems. Two sensors, each measuring the state of one of the two systems, compute and report their local state estimates to a central remote estimator, respectively. Due to the bandwidth constraint, at each time, only one of the sensors is allowed to communicate its estimate with the remote estimator. Upon receiving the data from the sensors, the remote estimator computes the minimum mean squared error estimate of each system's state. We provide an explicit construction of an optimal schedule, which is periodic (hence allows simple and efficient practical implementation) and minimizes the sum of the average estimation error covariance of each system.**

***Index Terms*—Communication constraint, Kalman filter, remote state estimation, sensor scheduling.**

I. Introduction

We consider scheduling two Gauss–Markov systems. Two sensors, each measuring the state of one of the systems respectively, are scheduled to send their local state estimates to a remote estimator for further processing. Due to the communication constraint (e.g., limited bandwidth), only one sensor is allowed to communicate with the remote estimator at each time.

Remote state estimation are found in many applications, such as in wireless sensor networks, networked industrial processes, smart grid, smart transportation, etc. [1]. If the communication media is perfect and has sufficient bandwidth, and guarantees reliable data flow, many existing tools such as the Kalman filter [2] can be used to estimate the state of a process. However, in many of the aforementioned applications, communication bandwidth is expensive, and different sensors and state estimators may be required to share the same network. In these cases, novel tools and methodologies for state estimation under communication constraints are needed. This research area has attracted much interest from different communities in recent years [3].

Mo *et al.* [4] considered sensor selection problems where a subset of sensors is to be selected at each time so that the network lifetime is maximized or the average error covariance at the estimator is minimized subject to limited energy budget constraint. By using convex relaxation techniques, a suboptimal sensor schedule is constructed. Shakeri *et al.* [5] considered sensor measurement scheduling subject to a finite cost constraint, where the measurement contributed a cost that is inversely proportional to its error covariance. They reduced the problem to a nonlinear optimization one with linear equality and inequality constraints. Vitus *et al.* [6] considered optimal sensor scheduling for a discrete-time system with multiple sensors, where only one sensor is allowed to take a measurement at each time. Chhetri *et al.* [7] proposed two sensor scheduling algorithms for a target tracking problem. Krishnamurthy [8] constructed algorithms for scheduling noisy sensors which measure the state of a single Markov chain. These algorithms aim to minimize a cost function of estimation errors and measurement costs. Chen *et al.* [9] presented transmission scheduling algorithms for maximizing the lifetime of a sensor network. The problem of dynamically selecting a group of sensors in their work was formulated as a stochastic shortest path Markov decision process. Cohen and Lesham [10] proposed a time-varying opportunistic protocol for network lifetime maximization when the sensors used are battery-powered and nonrechargeable. Chen *et al.* [11] considered the optimal transmission scheduling for maximizing the sensor network lifetime by utilizing the channel information. Dong *et al.* [12] considered the data retrieval problem in a 1-D sensor network. They also compared the performance of deterministic and random schedules. Zhang *et al.* [13] considered the infinite-horizon sensor scheduling problem for linear Gaussian processes with linear measurement functions. One of M sensors needs to be scheduled at each time to take a measurement of a single process so as to minimize the average estimation error covariance. The authors proved that the optimal estimation cost can be approximated arbitrarily closely by a periodic schedule with a finite period.

Most of the aforementioned works focused on a single process or target system. Multiple sensors are used to measure the state of the process, and only one or a subset of the sensors can report their measurement to a remote estimator at each time. In our paper, however, we focused on scheduling of two independent processes subject to communication constraint. We provide two motivational examples of the considered problem.

1) *Indoor environmental monitoring*: Consider measuring the temperature and humidity level inside an office using two sensors. At each time, either the sensor measuring the temperature or the sensor measuring the humidity reports its data to an access point (or remote estimator). The communication constraint is imposed to avoid potential data collision. Upon receiving the data from the two sensors, an estimate of the temperature and humidity is calculated.
2) *Target tracking*: Two sensors measuring two mobile targets (vehicles, missiles, etc.) need to report their readings to a central computational unit, which reads the data from one of them at each time. An estimate of the position of each target is calculated based on the sensor data received.

The main contribution of the paper is *the explicit construction of an optimal sensor schedule*, which is periodic, hence allowing efficient implementation in practice. To the best of our knowledge, such an explicit construction of an optimal sensor schedule is novel.

The remainder of the paper is organized as follows. Section II provides the mathematical problem setup. Some preliminary result is presented in Section III-A. The main result is then presented in Section III-B. Concluding remarks are given in the end.

Notations: $\mathbb{R}^n$ is the n-dimensional Euclidean space. $\mathbb{N}$ is the set of natural numbers. The nonnegative integer k is the time index. $\mathbb{S}_+^n$ is the set of n by n positive-semidefinite matrices. When $X \in \mathbb{S}_+^n$, we write $X \geq 0$; when X is positive definite, we write $X > 0$. For a matrix X, X' denotes its transpose. $\mathrm{Tr}[\cdot]$ denotes the trace of a matrix. $\mathbb{E}[\cdot]$ denotes the expectation of a random variable. For functions f_1, f_2, and f with appropriate domains, $f_1 \circ f_2(X) \triangleq f_1(f_2(X))$, $f^0(X) \triangleq X$, and $f^t(X) \triangleq f(f^{t-1}(X))$.

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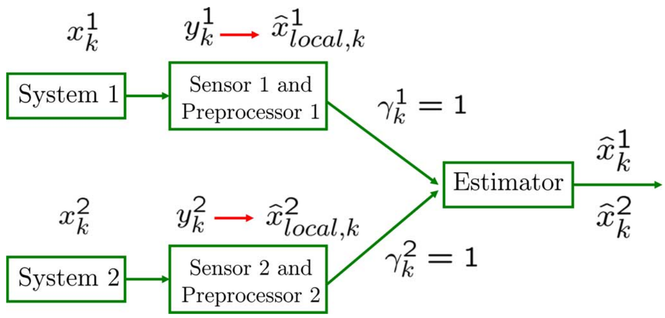

Fig. 1. System block diagram.

II. Problem Setup

Consider the following two Gauss–Markov systems (Fig. 1):

$$x_{k+1}^i = A_i x_k^i + w_k^i, \quad i = 1,2 \tag{1}$$
$$y_k^i = C_i x_k^i + v_k^i, \quad i = 1,2 \tag{2}$$

where $x_k^i \in \mathbb{R}^{n_i}$ is the state of the ith system at time k, $y_k^i \in \mathbb{R}^{m_i}$ is the measurement obtained by sensor i at time k, w_k^i's, v_k^i's and the initial system state x_0^i are mutually uncorrelated zero-mean Gaussian random variables with covariances $Q_i \geq 0$, $R_i > 0$, and $\Pi_i \geq 0$, respectively. A_i is unstable, and the pair $(A_i, \sqrt{Q}_i)$ is controllable and (A_i, C_i) is observable.

After obtaining y_k^i, the ith sensor (abbreviated as $\mathbf{s}_i$) computes[1] $\hat{x}_{\mathrm{local},k}^i$, the minimum mean-square error (MMSE) estimate of x_k^i using a Kalman filter as follows:

$$\begin{aligned}
\hat{x}_{\mathrm{local},k}^{i-} &= A_i \hat{x}_{\mathrm{local},k-1}^i \\
P_{\mathrm{local},k}^{i-} &= A_i P_{\mathrm{local},k-1}^i A_i' + Q_i \\
K_{\mathrm{local},k}^i &= P_{\mathrm{local},k}^{i-} C_i' \left[C_i P_{\mathrm{local},k}^{i-} C_i' + R_i \right]^{-1} \\
\hat{x}_{\mathrm{local},k}^i &= \hat{x}_{\mathrm{local},k}^{i-} + K_{\mathrm{local},k}^i \left[y_k^i - C_i \hat{x}_{\mathrm{local},k}^{i-} \right] \\
P_{\mathrm{local},k}^i &= \left[I - K_{\mathrm{local},k}^i C_i \right] P_{\mathrm{local},k}^{i-}
\end{aligned}$$

where the recursion starts from $\hat{x}_{\mathrm{local},0}^{i-} = 0$ and $P_{\mathrm{local},0}^{i-} = \Pi_i$.

Assume the communication bandwidth is limited and as a result, only one of the sensors is able to communicate with the remote estimator at each time step. Let $\gamma_k^i \in \{0,1\}$ indicate whether $\hat{x}_{\mathrm{local},k}^i$ is sent or not, i.e., if $\gamma_k^i = 1$, then $\hat{x}_{\mathrm{local},k}^i$ is sent to the remote estimator at time k; otherwise if $\gamma_k^i = 0$, $\hat{x}_{\mathrm{local},k}^i$ is not sent. Let $\theta = \{(\gamma_k^1, \gamma_k^2) : k = 0,1,2,\ldots\}$ be a sensor communication schedule that specifies the values of γ_k^1 and γ_k^2 for each k. We sometimes write γ_k^i as $\gamma_k^i(\theta)$ to indicate that the value of γ_k^i is determined by the schedule θ.

Define the information vector $I_k^i(\theta)$ as $I_k^i(\theta) \triangleq \{\gamma_0^i(\theta)\hat{x}_{\mathrm{local},0}^i, \ldots, \gamma_k^i(\theta)\hat{x}_{\mathrm{local},k}^i\}$, i.e., $I_k^i(\theta)$ contains all information the estimator has about the ith system at time k under the sensor communication schedule θ. Based on $I_k^i(\theta)$, the estimator computes $\hat{x}_k^i(\theta)$, the MMSE estimate of the state x_k^i, and its estimation error covariance $P_k^i(\theta)$ as

$$\begin{aligned}
\hat{x}_k^i(\theta) &= \mathbb{E}\left[x_k^i \,|\, I_k^i(\theta) \right], \\
P_k^i(\theta) &= \mathbb{E}\left[\left(x_k^i - \hat{x}_k^i \right) \left(x_k^i - \hat{x}_k^i \right)' \,|\, I_k^i(\theta) \right].
\end{aligned}$$

[1]Many sensors available in the market nowadays have on-board CPUs and a communication unit. The extra capabilities make them "smart" and be able to process and communicate their data with a remote state estimator.

Since $\hat{x}_{\mathrm{local},k}^i$ encodes $\{y_0^i, \ldots, y_k^i\}$, it is straightforward to show that the estimator should compute $\hat{x}_k^i(\theta)$ as follows[2]:

$$\hat{x}_k^i(\theta) = \begin{cases} \hat{x}_{\mathrm{local},k}^i, & \text{if } \gamma_k^i(\theta) = 1 \\ A_i \hat{x}_{k-1}^i(\theta), & \text{if } \gamma_k^i(\theta) = 0. \end{cases}$$

In other words, the remote estimator synchronizes its own estimate $\hat{x}_k^i$ with the received local estimate at the ith sensor if sensor i is scheduled to send data; otherwise, the remote estimator predicts x_k^i based on its previous optimal estimate $\hat{x}_{k-1}^i$. Consequently, $P_k^i(\theta)$ is computed as

$$P_k^i(\theta) = \begin{cases} P_{\mathrm{local},k}^i, & \text{if } \gamma_k^i(\theta) = 1 \\ A_i P_{k-1}^i(\theta) A_i' + Q_i, & \text{if } \gamma_k^i(\theta) = 0. \end{cases}$$

For a given schedule θ, define a cost function $J(\theta)$ associated with θ as follows:

$$J(\theta) \triangleq \limsup_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} \left(\mathrm{Tr}\left[P_k^1(\theta) \right] + \mathrm{Tr}\left[P_k^2(\theta) \right] \right). \tag{3}$$

Consider the set of all schedules Θ. This paper investigates the following problem[3]:

Problem 2.1:

$$\begin{aligned}
&\min_{\theta \in \Theta} J(\theta) \\
&\text{s.t.} \quad \gamma_k^1(\theta) + \gamma_k^2(\theta) = 1, \ \forall k \geq 0.
\end{aligned}$$

In other words, we wish to find a schedule θ which minimizes the sum of the trace of the average estimation error covariance of each system. Notice that the constraint is imposed by the finite communication bandwidth. A schedule $\theta \in \Theta$ is said to be optimal to Problem 2.1 if for any other $\bar{\theta} \in \Theta$, $J(\theta) \leq J(\bar{\theta})$.

For brevity, we will write $\gamma_k^i(\theta)$ as γ_k^i, $\hat{x}_k^i(\theta)$ as $\hat{x}_k^i$, and $P_k^i(\theta)$ as P_k^i, etc., when the underlying schedule θ is evident from the context. For any $t \in \mathbb{N}$, the notation $(\mathbf{s}_i)^t$ means sensor i is scheduled t times consecutively.

III. Main Results

A. Preliminaries

To simplify the notations and facilitate the analysis in subsequent sections, we define the functions h_i and g_i: $\mathbb{S}_+^{n_i} \to \mathbb{S}_+^{n_i}$ as follows:

$$\begin{aligned}
h_i(X) &\triangleq A_i X A_i' + Q_i \\
g_i(X) &\triangleq X - X C_i' [C_i X C_i' + R_i]^{-1} C_i X.
\end{aligned}$$

Then, one can verify that the local state estimation error covariance $P_{\mathrm{local},k}^i$ at the ith sensor at time k is given by

$$P_{\mathrm{local},k}^i = g_i \circ h_i \left(P_{\mathrm{local},k-1}^i \right).$$

Notice that applying h_i to $P_{\mathrm{local},k-1}^i$ corresponds to the time-update step in the Kalman filter; and applying g_i to $h_i(P_{\mathrm{local},k-1}^i)$ corresponds to the measurement-update step in the Kalman filter.

Since $(A_i, \sqrt{Q}_i)$ is controllable and (A_i, C_i) is observable, from standard Kalman filtering analysis (e.g., [2]), there exists a unique $P_i \geq 0$ such that

$$P_i = g_i \circ h_i(P_i) \tag{4}$$

[2]If $\hat{x}_{\mathrm{local},0}^i$ is not received, then $\hat{x}_0^i(\theta) = 0$ and $P_0^i(\theta) = \Pi_i$.

[3]It is not difficult to show that the optimal schedule remains the same if the constraint is replaced by a more general one: $\gamma_k^1(\theta) + \gamma_k^2(\theta) \leq 1, \forall k \geq 0$. The intuitive idea is that more communication between the sensors and the remote estimator leads to less estimation error. Thus the cost function is minimized only if $\gamma_k^1(\theta) + \gamma_k^2(\theta) = 1, \forall k \geq 0$.

which corresponds to the steady-state estimation error covariance of the Kalman filter at the ith sensor. Furthermore, $P^i_{\mathrm{local},k}$ converges to P_i exponentially fast. Since the cost function is over an infinite-horizon, we may ignore the transient period and assume the initial state covariance $\Pi_i = P_i$ without loss of generality. As a result $P^i_{\mathrm{local},k} = P_i, \forall k \geq 0$. Then, the estimation error covariance P^i_k at the remote estimator evolves according to

$$P^i_k = \begin{cases} P_i, & \text{if } \gamma^i_k = 1 \\ h_i\left(P^i_{k-1}\right), & \text{if } \gamma^i_k = 0. \end{cases}$$

Notice that $P^1_k = P_1$ and $P^2_k = P_2$ cannot hold simultaneously due to the communication constraint.

We now introduce the following important result on P_1 and P_2, which shall be used in deriving the main result in the next section. The proof is presented in the Appendix.

Lemma 3.1: For $i = 1, 2$, the following holds.

1) $h^{t_1}_i(P_i) \leq h^{t_2}_i(P_i)$ if $t_1 \leq t_2$, and $h^t_i(P_i) \neq h^{t+1}_i(P_i)$ for any $t \geq 0$.
2) For any $t \geq 1$,

$$\mathrm{Tr}[P_i] < \mathrm{Tr}[h_i(P_i)] < \cdots < \mathrm{Tr}\left[h^t_i(P_i)\right].$$

B. An Optimal Schedule

In this section, we present the main result of this paper: the explicit construction of an optimal sensor schedule to Problem 2.1. Without loss of generality, we limit the search of an optimal schedule to Problem 2.1 in a subset $\hat{\Theta} \subset \Theta$ that satisfy the following two conditions.

1) $\mathbf{s}_1$ is first scheduled at $k = 0$.
2) A schedule $\theta \in \hat{\Theta}$ can be presented as

$$\left(\mathbf{s}_1^{l_{11}}\mathbf{s}_2^{l_{21}}\right)\left(\mathbf{s}_1^{l_{12}}\mathbf{s}_2^{l_{22}}\right)\cdots\left(\mathbf{s}_1^{l_{1t}}\mathbf{s}_2^{l_{2t}}\right)\cdots \tag{5}$$

for some $l_{1t} \geq 1, l_{2t} \geq 1, t = 1, 2, \ldots$, i.e., under θ, $\mathbf{s}_1$ is first scheduled for l_{11} times, then $\mathbf{s}_2$ is scheduled for l_{21} times; after that, $\mathbf{s}_1$ is scheduled for l_{12} times, and $\mathbf{s}_2$ is scheduled for l_{22} times; and so on, and so forth.

If an optimal schedule θ does not satisfy the first condition and schedules $\mathbf{s}_2$ first for a consecutive T times before scheduling $\mathbf{s}_1$, then we can construct a schedule θ' by letting $\gamma^i_k(\theta') = \gamma^i_{k+T}(\theta), k = 0, 1, 2, \ldots$. It is straightforward to show that $J(\theta) = J(\theta')$, thus θ' is also optimal that satisfies the first condition.

The second condition is derived from the first one and the fact that $\mathbf{s}_1$ or $\mathbf{s}_2$ cannot be scheduled consecutively for an infinite number as P^2_k or P^1_k will diverge otherwise. Now consider any finite $t \geq 1$. Let

$$T_t = \sum_{i=1}^{t}(l_{1i} + l_{2i}).$$

Define $J_t(\theta)$ for a $\theta \in \hat{\Theta}$ as

$$J_t(\theta) = \frac{1}{T_t}\sum_{k=0}^{T_t-1}\left(\mathrm{Tr}\left[P^1_k(\theta)\right] + \mathrm{Tr}\left[P^2_k(\theta)\right]\right). \tag{6}$$

Clearly, $J(\theta) = \limsup_{t\to\infty} J_t(\theta)$. Further define $q_1(l)$ and $q_2(l)$ as follows:

$$q_1(l) = \sum_{i=1}^{l}\left(\mathrm{Tr}[P_1] + \mathrm{Tr}\left[h^l_2(P_2)\right]\right)$$

$$q_2(l) = \sum_{i=1}^{l}\left(\mathrm{Tr}[P_2] + \mathrm{Tr}\left[h^l_1(P_1)\right]\right).$$

Then, one easily verifies using the preliminaries from the previous section that $J_t(\theta)$ with θ presented in the form of (5) is given in a closed-form by

$$J_t(\theta) = \frac{1}{T_t}\sum_{i=1}^{t}[q_1(l_{1i}) + q_2(l_{2i})], \tag{7}$$

from which we obtain the following:

$$\begin{aligned} J_t(\theta) &= \sum_{i=1}^{t}\frac{1}{T_t}[q_1(l_{1i}) + q_2(l_{2i})] \\ &= \sum_{i=1}^{t}\frac{l_{1i}+l_{2i}}{T_t}\frac{1}{l_{1i}+l_{2i}}[q_1(l_{1i}) + q_2(l_{2i})]. \end{aligned}$$

Since $\sum_{i=1}^{t}\frac{l_{1i}+l_{2i}}{T_t} = 1$, $J_t(\theta)$ is a convex combination of the t quantities $\frac{1}{l_{1i}+l_{2i}}[q_1(l_{1i}) + q_2(l_{2i})], i = 1, \ldots, t$. Thus $J_t(\theta)$ has to be no less than the minimum of these t quantities, i.e.,

$$\begin{aligned} J_t(\theta) &\geq \min_{(l_{1i},l_{2i})}\frac{1}{l_{1i}+l_{2i}}[q_1(l_{1i}) + q_2(l_{2i})] \\ &= \min_{(k_1,k_2)}\frac{1}{k_1+k_2}[q_1(k_1) + q_2(k_2)] \end{aligned} \tag{8}$$

where $k_1 \geq 1$ and $k_2 \geq 1$. Since (8) holds for any $t \geq 1$, one immediately obtains that

$$J(\theta) \geq \min_{(k_1,k_2)}\frac{1}{k_1+k_2}[q_1(k_1) + q_2(k_2)]. \tag{9}$$

We now present one of the main results of this paper in the following theorem.

Theorem 3.2: Let $k_1 = k_1^\star \geq 1$ and $k_2 = k_2^\star \geq 1$ minimizes $f(k_1, k_2)$ which is defined as

$$f(k_1, k_2) \triangleq \frac{1}{k_1+k_2}[q_1(k_1) + q_2(k_2)]. \tag{10}$$

Let $\theta^\star$ be a periodic schedule with period $k_1^\star + k_2^\star$ which schedules $\mathbf{s}_1$ and $\mathbf{s}_2$ in the first period as $\mathbf{s}_1^{k_1^\star}\mathbf{s}_2^{k_2^\star}$. Then $\theta^\star$ is an optimal schedule to Problem 2.1.

Proof: The cost of $\theta^\star$ is easily seen to be given by

$$J(\theta^\star) = \frac{1}{k_1^\star + k_2^\star}[q_1(k_1^\star) + q_2(k_2^\star)].$$

Thus, from (9), for a general $\theta \in \hat{\Theta}$,

$$\begin{aligned} J(\theta) &\geq \min_{(k_1,k_2)}\frac{1}{k_1+k_2}[q_1(k_1) + q_2(k_2)] \\ &= \frac{1}{k_1^\star + k_2^\star}[q_1(k_1^\star) + q_2(k_2^\star)) \\ &= J(\theta^\star) \end{aligned}$$

which establishes the optimality of θ. ∎

Now, we have shown that the periodic schedule $\theta^\star$ is optimal and our remaining task is to find the exact values of $k_1^\star$ and $k_2^\star$ to allow efficient implementation in practice. First, we have the following result.

Lemma 3.3: Let $k_1 > 2$ and $k_2 > 2$. If

$$\frac{1}{k_1}q_1(k_1) \leq \frac{1}{k_2}q_2(k_2), \tag{11}$$

then $f(k_1, k_2 - 1) < f(k_1, k_2)$; otherwise $f(k_1 - 1, k_2) < f(k_1, k_2)$.

Proof: To simplify the notations, let us define

$$\alpha = \frac{k_1}{k_1+k_2}, \delta = \frac{k_1}{k_1+k_2-1},$$
$$\text{and}$$
$$x = \frac{1}{k_1}q_1(k_1), y = \frac{1}{k_2}q_2(k_2), y' = \frac{1}{k_2-1}q_2(k_2-1).$$

Assume (11) holds, i.e., $y \geq x$. From Lemma 3.1, $y > y' > 0$. Furthermore, $1 > \delta > \alpha > 0$. From these inequalities, we obtain

$$\begin{aligned} f(k_1,k_2) - f(k_1,k_2-1) &= \alpha x + (1-\alpha)y - [\delta x + (1-\delta)y'] \\ &> \alpha x + (1-\alpha)y - [\delta x + (1-\delta)y] \\ &= (\delta - \alpha)(y-x) \geq 0. \end{aligned}$$

The remaining part of the lemma can be proved similarly. ∎

From Lemma 3.3, it is straightforward to see that $k_1^\star = 1$ or $k_2^\star = 1$. If

$$\mathrm{Tr}[P_1] + \mathrm{Tr}[h_2(P_2)] \geq \mathrm{Tr}[P_2] + \mathrm{Tr}[h_1(P_1)], \tag{12}$$

then $k_1^\star = 1$; otherwise $k_2^\star = 1$. Notice that $k_1^\star = k_2^\star = 1$ if (12) becomes an equality.

Without loss of generality, assume (12) holds and $k_1^\star = 1$. We would like to find $k_2^\star$. Define $k_{\max}$ as

$$k_{\max} \triangleq \min_{l \geq 1} \left\{ l : \frac{1}{l} q_2(l) \geq q_1(1) \right\}. \tag{13}$$

Then, we have the following result.

Theorem 3.4: $k_2^\star$ satisfies

$$1 \leq k_2^\star \leq k_{\max}. \tag{14}$$

Proof: It suffices to prove that if $k_2 > k_{\max}$, then $f(1,k_2) > f(1,k_{\max})$. We redefine

$$\alpha = \frac{1}{1+k_{\max}}, \quad \delta = \frac{1}{1+k_2},$$
and
$$x = q_1(1), \quad y = \frac{q_2(k_{\max})}{k_{\max}}, \quad y' = \frac{q_2(k_2)}{k_2}.$$

Now, $1 > \alpha > \delta > 0$ and $y' > y > 0$. Furthermore, from the definition of $k_{\max}$ in (13), $y \geq x$. Thus, we have

$$\begin{aligned} f(1,k_2) - f(1,k_{\max}) &= \delta x + (1-\delta)y' - \alpha x - (1-\alpha)y \\ &> \delta x + (1-\delta)y - \alpha x - (1-\alpha)y \\ &= (\alpha - \delta)(y-x) \geq 0. \end{aligned}$$

∎

Since A is unstable, $\mathrm{Tr}[h_1^t(P_1)]$ increases exponentially in t. As a result, $k_{\max}$ can be found efficiently. After $k_{\max}$ is found, we can evaluate $k_2^\star$ from (10) and (14), which needs to compare $k_{\max}$ numbers $\{f(1,1), f(1,2), \ldots, f(1,k_{\max})\}$ and find out the minimum number. The complexity is only $\mathcal{O}(k_{\max})$. The following result, however, states that we can find the optimal $k_2^\star$ in less than $k_{\max}$ steps.

Proposition 3.5: $k_2^\star$ defined by (14) can be calculated as

$$k_2^\star = \min_{1 \leq k_2 \leq k_{\max}} \{k_2 : f(1,k_2+1) \geq f(1,k_2)\}. \tag{15}$$

Proof: Note that it suffices to prove if $f(1,k_2+1) \geq f(1,k_2)$, then $f(1,k_2+i) > f(1,k_2)$ for any $i \geq 2$. Now

$$\begin{aligned} & f(1,k_2+1) \geq f(1,k_2) \\ &\Longrightarrow (k_2+2)f(1,k_2+1) \geq (k_2+1)f(1,k_2) + f(1,k_2) \\ &\Longrightarrow \mathrm{Tr}\left[h_1^{k_2+1}(P_1)\right] + \mathrm{Tr}[P_2] \geq f(1,k_2). \end{aligned} \tag{16}$$

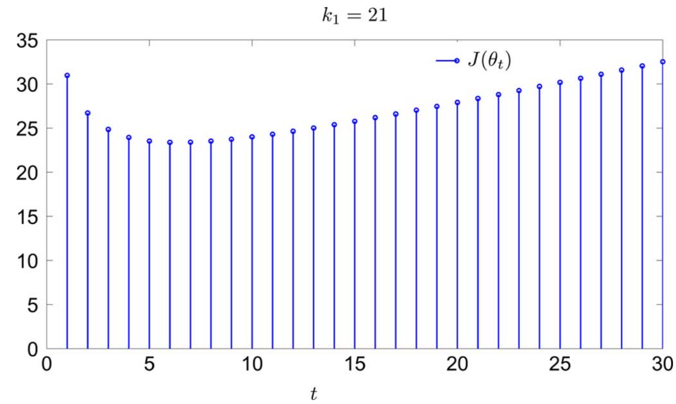

Fig. 2. $J(\theta_t)$ as a function of θ_t.

Therefore

$$\begin{aligned} & f(1,k_2+i) > f(1,k_2) \\ &\Longleftrightarrow (k_2+i+1)f(1,k_2+i) > (k_2+1)f(1,k_2) + if(1,k_2) \\ &\Longleftrightarrow \mathrm{Tr}\left[h_1^{k_2+i}(P_1)\right] + \cdots + \mathrm{Tr}\left[h_1^{k_2+1}(P_1)\right] + i\mathrm{Tr}[P_2] > if(1,k_2). \end{aligned}$$

From Lemma 3.1 and (16), the last inequality holds as

$$\begin{aligned} & \mathrm{Tr}\left[h_1^{k_2+i}(P_1)\right] + \cdots + \mathrm{Tr}\left[h_1^{k_2+1}(P_1)\right] + i\mathrm{Tr}[P_2] \\ &\qquad > i\left(\mathrm{Tr}\left[h_1^{k_2+1}(P_1)\right] + \mathrm{Tr}[P_2]\right) \geq if(1,k_2). \end{aligned}$$

Thus, the proof is completed. ∎

C. Example

Consider the following two second-order systems with

$$A_1 = \begin{bmatrix} 1 & 0.001 \\ 0 & 0.9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0.9 \\ 0 & 2 \end{bmatrix},$$
$$C_1 = [1 \quad 1], \quad C_2 = [1 \quad 0],$$

and $Q_1 = I, Q_2 = I, R_1 = 0.1, R_2 = 0.5$. It is found that $k_1^\star = 1$ as the inequality in (12) holds. According to Proposition 3.5, we evaluated the first few $f(1,k_2)$ for $k_2 = 1,\ldots,7$ whose values are given by $f(1,1) = 30.9884, f(1,2) = 26.7102, f(1,3) = 24.8499, f(1,4) = 23.9522, f(1,5) = 23.5328, f(1,6) = 23.3847, f(1,7) = 23.4046$. Thus, we found $k_2^\star = 6$. The cost function $J(\theta_{k_2})$ for different k_2's are plotted in Fig. 2, where θ_{k_2} stands for the periodic schedule which schedules $\mathbf{s}_1$ and $\mathbf{s}_2$ in a period of k_2+1 as $\mathbf{s}_1\mathbf{s}_2^{k_2}$. The plot is consistent with our result in Proposition 3.5.

IV. Conclusion

In this paper, we consider scheduling of two sensors each reporting its own local state estimate to a remote estimator. The two sensors measure two independent Gauss–Markov systems respectively. Due to the communication bandwidth constraint, only one of the sensors is allowed to communicate with the remote estimator. We provide an explicit construction of an optimal sensor schedule. Future work will be extending the results developed in this paper to multiple sensors schedule problems.

Appendix

Proof to Lemma 3.1: 1) From the definition of g_i, for any $X \geq 0$,

$$g_i(X) = X - XC_i'[C_iXC_i' + R_i]^{-1}C_iX \leq X.$$

Therefore

$$P_i = g_i \circ h_i(P_i) \leq h(P_i). \tag{17}$$

Since $h_i(X) \leq h_i(Y)$ for any $Y \geq X \geq 0$, by applying h_i repeatedly on both sides of (17), we get

$$P_i \leq h_i(P_i) \leq \cdots \leq h_i^t(P_i), \quad \forall t \geq 1.$$

Now assume there exists $t \geq 0$ such that

$$h_i^t(P_i) = h_i^{t+1}(P_i). \tag{18}$$

Then from (18), we obtain the following $h_i^t(P_i) = h_i^{t+l}(P_i), \forall l \geq 1$. Recall that n_i is the state dimension of the ith system. Consider $l = n_i$. Then

$$\begin{aligned} h_i^t(P_i) &= h_i^{n_i}\left(h_i^t(P_i)\right) \\ &= A_i^{n_i} h_i^t(P_i)(A_i')^{n_i} + \sum_{j=0}^{n_i-1} A_i^j Q_i (A_i')^j. \end{aligned}$$

As $(A_i, \sqrt{Q_i})$ is controllable, the controllability matrix W_c^i of the ith system, which is given by

$$W_c^i = \left[\sqrt{Q_i} \quad A_i\sqrt{Q_i} \quad \cdots \quad A_i^{n_i-1}\sqrt{Q_i}\right]$$

has rank n_i. This leads to the fact that

$$\sum_{j=0}^{n_i-1} A_i^j Q_i (A_i')^j = \left(W_c^i\right)\left(W_c^i\right)' > 0,$$

thus $h_i^t(P_i) > 0$. Now considering $l = 1$, we have $h_i^t(P_i) = h_i(h_i^t(P_i))$. Hence, $h_i^t(P_i) > 0$ is a positive-definite solution to the Lyapunov equation $X = h_i(X) = A_i X A_i' + Q_i$. Again from the fact that $(A_i, \sqrt{Q_i})$ is controllable, we conclude that A is stable, which contradicts with the assumption that A is unstable. This completes the proof for the first part.

2) From the first part, we immediately have $\mathrm{Tr}[P_i] \leq \mathrm{Tr}[h_i(P_i)] \leq \cdots \leq \mathrm{Tr}[h_i^t(P_i)]$. Suppose there exists $t \geq 0$ such that $\mathrm{Tr}[h_i^t(P_i)] = \mathrm{Tr}[h_i^{t+1}(P_i)]$. Then, on the one hand, $h_i^{t+1}(P_i) - h_i^t(P_i) \geq 0$ from the first part. On the other hand, $\mathrm{Tr}[h_i^{t+1}(P_i) - h_i^t(P_i)] = 0$. Therefore, we conclude that the eigenvalues of the positive semi-definite matrix $h_i^{t+1}(P_i) - h_i^t(P_i)$ are all zeros, which implies that $h_i^{t+1}(P_i) - h_i^t(P_i) = 0$. Thus $h_i^{t+1}(P_i) = h_i^t(P_i)$, which contradicts with the first part. Therefore, for all $t \geq 0$, $\mathrm{Tr}[h_i^t(P_i)] < \mathrm{Tr}[h_i^{t+1}(P_i)]$. This completes the proof for the second part.

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Single-Transmission Distributed Detection via Order Statistics

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***Abstract*—Consider a sensor network made of remote nodes connected to a common fusion center. In a recent work, Blum and Sadler proposed the idea of ordered transmissions—sensors with more informative measurements deliver their messages first—and they proved that optimal detection performance can be achieved using only a subset of the measurements available to the system. Taking to one extreme this approach, we show that using only one transmission the detection error can be made as small as desired, provided that the network size n is large enough. Indeed, we design a distributed detection scheme and prove its asymptotic consistency with respect to n, when the decision is made using just one—but the best—out of n collected samples.**

***Index Terms*—Consistent detection, ordered transmissions, wireless sensor network.**

I. INTRODUCTION

Consider a distributed detection system, e.g., a wireless sensor network (WSN), designed to solve a binary hypothesis test, in which the remote nodes collect their measurements and send messages to the fusion center (FC), at which the final decision is made. Exploiting the

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