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Deterministic Sensor Data Scheduling Under Limited Communication Resource

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***Abstract*—We consider finite time-horizon sensor data scheduling under limited communication resource. A sensor can only send d of its measurement data to a remote estimator within a time-horizon $T \gg d$. When use the terminal estimation error covariance of the estimator as a performance metric, we provide an explicit form of the optimal data schedule; when use the average estimation error covariance as a performance metric, we provide a necessary condition for a schedule to be optimal for a general T. When T has a special form, the necessary condition allows us to construct an explicit optimal data schedule.**

***Index Terms*—Communication constraint, Kalman filter, remote state estimation, sensor data scheduling.**

I. Introduction

Networked sensing, estimation, and control systems have attracted much attention over the past decade [1], thanks to the recent advances in sensor and communication technology. Remote state estimation and information processing have a wide range of applications such as in environmental monitoring, body sensor network, vehicle navigation, industrial process, smart grid, etc.

In many of the aforementioned applications, sensors that collect physical data of interest may be battery-powered, which means that the energy for the sensors to communicate with remote data processors is limited. Network bandwidth is often a scarce resource and could be limited as well. Therefore, a sensor may not be able to communicate with a remote data processor at each time. These practical constraints require an appropriate data scheduling algorithm (referred simply as a schedule) to balance the limited communication resource and the performance of the remote data processor.

Sensor data scheduling has been a hot topic due to its practical importance as well as the technical challenges involved. Savage and La Scala [2] considered a sensor measurement scheduling problem. Within a finite time horizon N, to minimize the terminal estimation error covariance, they provided the optimal schedule under the constraint that only $n < N$ measurements could be taken. Shakeri *et al.* [3] considered sensor measurement scheduling subject to a fixed cost constraint, where the measurement contributed a cost inversely proportional to its error covariance. They reduced the problem to a nonlinear optimization one with linear equality and inequality constraints. Vitus *et al.* [4] considered optimal sensor scheduling of a discrete-time system with multiple sensors where only one sensor is allowed to take a measurement. Arai *et al.* [5] considered a similar problem, and proposed a fast sensor scheduling algorithm. Chhetri *et al.* [6] proposed two sensor scheduling algorithms for a target tracking problem. Krishnamurthy [7] constructed algorithms for scheduling noisy sensors which measure the state of a single Markov chain. These algorithms aim to minimize a cost function of estimation errors and measurement costs. Cohen and Lesham [8] proposed a time-varying opportunistic protocol for network lifetime maximization when the sensors used are battery-powered and non-rechargeable. Chen *et al.* [9] considered the optimal transmission scheduling for maximizing the sensor network lifetime by utilizing the channel information. Dong *et al.* [10] considered the data retrieval problem in a 1-D sensor network. They also compared the performance of deterministic and random schedules.

The main contribution of this paper is the construction of optimal schedules of sensor communication times under the constraint that the sensor can only communicate with the remote state estimator d times within a time-horizon $T \gg d$. When use the terminal estimation error covariance of the estimator as a performance metric, we provide an explicit form of the optimal data schedule; when use the average estimation error covariance as a performance metric, we provide a necessary condition for a schedule to be optimal for a general T. When T has a special form, the necessary condition allows us to construct an explicit optimal data schedule. Notice that the set of all possible deterministic schedules contains $\binom{T}{d}$ elements, thus finding an optimal schedule is in general a challenging task. To the best of our knowledge, explicit form of the optimal schedule has only been obtained in literature for the terminal error performance metric for a special class of systems (see Remark 3.2).

The remainder of this paper is organized as follows. Section II gives the problem description. Section III presents the optimal schedule which minimizes the terminal estimation error covariance (Theorem 3.1). Section IV gives a necessary condition on the optimal schedule which minimizes the average estimation error covariance for a general T (Theorem 4.2). Such a necessary condition allows us to construct the optimal schedule when T takes a special form (Theorem 4.3). Some examples and concluding remarks are provided in the end.

Notations: $\mathbb{N}$ is the set of natural numbers. $\mathbb{Z}$ is the set of non-negative integers. $k \in \mathbb{Z}$ is the time index. $\mathbb{R}^n$ is the n-dimensional Euclidian space. $\mathbb{R}_+$ is the set of non-negative real numbers. $\mathbb{S}_+^n$ is the set of all n by n positive semi-definite matrices. For functions $f,\ f_1,\ f_2$ with appropriate domains, $f_1 f_2(x)$ stands for the function composition $f_1\Big(f_2(x)\Big)$, and $f^t(x) \triangleq f\Big(f^{t-1}(x)\Big)$ with $f^0(x) \triangleq x$.

II. Problem Setup

Consider the following first-order Gauss–Markov system (Fig. 1)

$$x_{k+1} = a x_k + w_k \tag{1}$$

$$y_k = c x_k + v_k \tag{2}$$

where x_k is the system state at time k, y_k is the measurement obtained by a sensor, w_k's, v_k's and the initial state x_0 are mutually uncorrelated zero-mean Gaussian random variables with covariances $q > 0,\ r > 0$ and $\pi_0 > 0$ respectively. Assume $|a| \geq 1$ and $c \neq 0$.

After obtaining the measurement, the sensor decides if it will transmit y_k to a remote estimator for further processing. The estimator calculates the minimum mean-squared error estimate of the state x_k in (1) based upon all measurement data it receives by time k. It is well known that such estimate and its associated error covariance can

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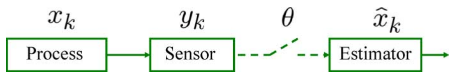

Fig. 1. Sensor measurement scheduling diagram.

be computed via a Kalman filter [11]. Denote the predicted estimate before receiving y_k as $\hat{x}_{k|k-1}$, which is called the *a priori* estimate, and the associated error covariance as $p_{k|k-1}$. After receiving y_k, denote the estimate as $\hat{x}_{k|k}$ and the associated error covariance as $p_{k|k}$, which are called the *a posteriori* estimate and error covariance. If y_k is received, the computation is as follows:

$$\begin{aligned}\hat{x}_{k|k-1} &= a\hat{x}_{k-1|k-1}\\ p_{k|k-1} &= a^2 p_{k-1|k-1} + q\\ l_k &= \frac{p_{k|k-1}c}{c^2 p_{k|k-1} + r}\\ \hat{x}_{k|k} &= \hat{x}_{k|k-1} + l_k(y_k - c\hat{x}_{k|k-1})\\ p_{k|k} &= (1 - l_k c)p_{k|k-1}\end{aligned}$$

where the recursion starts from $\hat{x}_{0|0} = 0$ and $p_{0|0} = \pi_0$. Otherwise if y_k is not received, the *a posteriori* simply equals the *a priori* one, i.e.,

$$\begin{aligned}\hat{x}_{k|k} &= a\hat{x}_{k-1|k-1}\\ p_{k|k} &= a^2 p_{k-1|k-1} + q.\end{aligned}$$

For brevity, we write $\hat{x}_{k|k}$ as $\hat{x}_k$ and $p_{k|k}$ as p_k for the remainder of the paper.

Assume the sensor can only communicate d times with the remote estimator within a time horizon $T \gg d$. The constraint could be imposed by the limited transmission energy at the sensor or finite bandwidth of the network.

Define a deterministic schedule θ as a set of transmission times: $\{k_1, k_2, \ldots, k_d\}$ with $1 \le k_1 < \cdots < k_d \le T$, i.e., k_i is a time when y_{k_i} is transmitted to the estimator. Clearly, $\hat{x}_k$ and p_k depends explicitly on the schedule θ being used. Thus, we will write them as $\hat{x}_k(\theta)$ and $p_k(\theta)$ if we want to highlight this dependence. Let Θ be the set of all deterministic schedules, which contains $\binom{T}{d}$ elements. In general, the estimator has different performance corresponding to different schedules. In this paper, we consider two types of performance metrics, the terminal error covariance and the average error covariance of the estimator, each corresponding to a certain class of practical applications. In other words, we wish to find a schedule $\theta \in \Theta$ such that 1) $p_T(\theta)$ is minimized and 2) $\sum_{k=1}^{T} p_k(\theta)$ is minimized. The two cases will be considered separately in the next two sections. To simplify notations in subsequent sections, let us introduce the functions $h, g : \mathbb{R}_+ \to \mathbb{R}_+$ as follows:

$$h(x) \triangleq a^2 x + q \tag{3}$$

$$g(x) \triangleq a^2 x + q - \frac{c^2(a^2x+q)^2}{c^2(a^2x+q)+r}. \tag{4}$$

From their definitions, it is straightforward to verify that both $h(x)$ and $g(x)$ are monotonically increasing functions for any $x \in \mathbb{R}_+$ and $g(x) \le h(x)$. Furthermore, we have the following result on h and g.

Lemma 2.1: When $|a| \ge 1$ and $c \ne 0$,

$$gh(x) < hg(x),\ \forall x \in \mathbb{R}_+. \tag{5}$$

Proof: See the Appendix. ■

Using the functions h and g, one can verify that p_k is given recursively as

$$p_k = \begin{cases} g(p_{k-1}), & \text{if } y_k \text{ is received}\\ h(p_{k-1}), & \text{if } y_k \text{ is not received.}\end{cases} \tag{6}$$

III. Optimal Schedule for Minimizing the Terminal Error Covariance

In this section we will find an optimal schedule θ that minimizes $p_T(\theta)$. Intuitively since the cost function is the terminal error covariance, the sensor should spend all its communications with the estimator in the last d time steps. Indeed, this intuition holds, which is captured in the following theorem.

Theorem 3.1: The optimal schedule $\theta^* = \{k_1^\star, k_2^\star, \ldots, k_d^\star\}$ which minimizes $p_T(\theta)$ is given by $k_i^\star = T - d + i, i = 1, 2, \ldots, d$.

Proof: From the definition of θ^*, $p_T(\theta^*) = g^d h^{T-d}(p_0)$. To prove θ^* is optimal, consider any $\theta = \{k_1, \ldots, k_d\}$ that is different from θ^*, and we shall show that $p_T(\theta^*) < p_T(\theta)$. This holds as

$$\begin{aligned}&p_T(\theta)\\ &= h^{T-k_d} g h^{k_d - k_{d-1}-1} g h^{k_{d-1}-k_{d-2}-1} \cdots g h^{k_2-k_1-1} g h^{k_1-1}(p_0)\\ &\ge g h^{T-k_{d-1}-1} g h^{k_{d-1}-k_{d-2}-1} \cdots g h^{k_2-k_1-1} g h^{k_1-1}(p_0)\\ &\ge g^2 h^{T-k_{d-2}-2} \cdots g h^{k_2-k_1-1} g h^{k_1-1}(p_0)\\ &\vdots\\ &\ge g^d h^{T-d}(p_0) = p_T(\theta^*)\end{aligned}$$

where the inequalities are from (5) and that both h and g are increasing functions. Since $\theta \ne \theta^*$, from (5), at least one of the above inequalities is a strict inequality. Hence, we conclude that $p_T(\theta^*) < p_T(\theta)$. ■

Remark 3.2: This result is the same as part of the work Savage and Scala did in [2]. However, [2] only discussed the $a = c = 1$ case. Theorem 3.1 covers the more general case $|a| \ge 1$ and $c \ne 0$.

Remark 3.3: For a general nth order Gauss–Markov system

$$\begin{aligned}x_{k+1} &= Ax_k + w_k\\ y_k &= Cx_k + v_k\end{aligned}$$

with w_k and v_k being zero-mean Gaussian random vectors having covariance matrix $Q \ge 0$ and $R \ge 0$, respectively, Theorem 3.1 still holds if

$$gh(X) \le hg(X),\ \text{for } X \ge 0 \tag{7}$$

where h and g (now $\mathbb{S}_+^n \to \mathbb{S}_+^n$) are redefined as

$$\begin{aligned}h(X) &\triangleq AXA' + Q\\ g(X) &\triangleq h(X) - h(X)C'[Ch(X)C' + R]^{-1}Ch(X).\end{aligned}$$

For example, if $Q \ge (C'R^{-1}C)^{-1}$, then it is not difficult to verify that (7) holds.

IV. Optimal Schedule for Minimizing the Average Error Covariance

In this section we wish to find an optimal schedule θ that minimizes the cost $J(\theta)$ defined as

$$J(\theta) \triangleq \sum_{k=1}^{T} p_k(\theta). \tag{8}$$

Intuitively, since the cost function is the average error covariance, the sensor should spend the d communications with the estimator *as uniform as possible* within T. However, compared with the terminal error case, this is quite challenging to prove. In fact, it may not always hold. Nevertheless, with some additional assumptions, we can prove this intuition does hold. The following two conditions are assumed in this section: 1) the initial condition $p_0 \in [g_{\min}, g_{\max})$; 2) $\frac{r}{c^2} \le q$, where $g_{\min} \triangleq \min_{x\in\mathbb{R}_+} g(x) = g(0) = \frac{q\frac{r}{c^2}}{q+\frac{r}{c^2}} \ge \frac{r}{2c^2}$ and $g_{\max} \triangleq \max_{x\in\mathbb{R}_+} g(x) = g(\infty) = \frac{r}{c^2}$. The first assumption is to facilitate our discussion and can be extended to general case. The second assumption means we require the scaled measurement noise has a smaller covariance than the system noise, i.e., the sensor provides a relatively accurate measurement. We start with the simple case $d = 1$ where an explicit form of the optimal schedule is provided. We will then cover the more general case. When $d = 1$, we have the following result.

Theorem 4.1: Denote the optimal schedule as $\theta^\star = \{k^\star\}$. For $T \in \mathbb{N}$, if $T = 2m + 1$ for some $m \in \mathbb{Z}$, then $k^\star = m + 1$. If $T = 2m$, then $k^\star = m$ or $k^\star = m + 1$.

Proof: See the Appendix. ■

For a general d, we have the following necessary condition on the optimality of θ.

Theorem 4.2: Let $T = (d + 1)M + L, 0 \le L \le d$, and denote $\omega_1 = k_1 - 1, \omega_i = k_i - k_{i-1} - 1,\ i = 2, \ldots, d$ and $\omega_{d+1} = T - k_d$. If θ is optimal, then $\omega_j = M$ or $M - 1,$ for all j. Furthermore, denote the number of interval with length $M - 1$ and M as N_{M-1} and N_M. Then $N_{M-1} = d - L$ and $N_M = L + 1$.

Proof: See the Appendix. ■

Theorem 4.2 contains an important fact, which is stated in the following theorem.

Theorem 4.3: When $T = (d+1)M - 1$, then the optimal schedule $\theta^* = \{k_1^*, k_2^*, \ldots, k_d^*\}$ is given by $k_i^* = iM,\ i = 1, 2, \ldots, d$, i.e., the communication times are separated as uniformly as possible.

Proof: From Theorem 4.2, the length of intervals can only be $M - 1$ and $M - 2$. Furthermore, $N_{M-1} + N_{M-2} = d + 1$ and $N_{M-1}(M - 1) + N_{M-2}(M - 2) + d = T$, from which we obtain $N_{M-1} = d + 1$ and $N_{M-2} = 0$. ■

Remark 4.4: When taking the terminal error as the cost function, in Remark 3.3, we give a sufficient condition (7) such that Theorem 3.1 still holds for a higher order Gauss–Markov system. Unfortunately, when taking the average error as the cost function, finding an optimal schedule for a general higher order Gauss–Markov system is challenging and remains an open problem.

Remark 4.5: In this paper, we mainly focused on one sensor scenario. In many remote estimation examples, a network of sensors may be used to gather the state information. When the network bandwidth is limited, only a subset of the sensors can communicate their data with a remote estimator at each time. At the same time, each sensor may be battery-powered and can only communicate with the remote estimator for a limited times. Following the same framework developed in this paper, we can ask when and which sensor should send their data such that the estimation error at the remote estimator is minimized. Mathematically, consider the following system:

$$\begin{aligned} x_{k+1} &= a x_k + w_k \\ y_k^i &= c_i x_k + v_k^i,\ i = 1, 2, \ldots, N \end{aligned}$$

where x_k is the system state at time k, y_k^i is the measurement obtained by the ith sensor, w_k's, v_k^i's, and the initial state x_0 are mutually uncorrelated zero-mean Gaussian random variables with covariances $q > 0,\ r_i > 0$ and $\pi_0 > 0$, respectively.

Define a schedule θ as a set of communication control variables $\{\gamma_k^1, \gamma_k^2, \ldots, \gamma_k^N\}, k = 1, 2, \ldots, T$, where $\gamma_k^i = 1$ or 0 and $\gamma_k^i = 1$ represents that the ith sensor will send its measurement to the remote estimator at time k. Upon receiving the measurements from the sensors, the remote estimator computes the state estimator $\hat{x}_k$ and the associated error covariance p_k as follows (known as the Kalman filter in the *information form* [12]):

$$\begin{aligned} \hat{x}_{k|k-1} &= a\hat{x}_{k-1|k-1} \\ p_{k|k-1} &= a^2 p_{k-1|k-1} + q \\ (p_{k|k})^{-1} &= (p_{k|k-1})^{-1} + \sum_{i:\gamma_k^i=1} \frac{c_i^2}{r_i} \\ \hat{x}_{k|k} &= p_{k|k}\left[(p_{k|k-1})^{-1}\hat{x}_{k|k-1} + \sum_{i:\gamma_k^i=1} \frac{c_i}{r_i} y_k^i\right] \end{aligned}$$

where the recursion starts from $\hat{x}_{0|0} = 0$ and $p_{0|0} = \pi_0$.

We can then cast the problem of optimal sensor scheduling in a multisensor scenario as follows:

$$\begin{aligned} \min \quad & J(\theta) \triangleq \sum_{k=1}^{T} p_k(\theta) \\ \text{s.t.} \quad & \sum_{i=1}^{N} \gamma_k^i \le b,\ k = 1, 2, \ldots, T \\ & \sum_{k=1}^{T} \gamma_k^i \le d_i,\ i = 1, 2, \ldots, N \end{aligned}$$

where T is the considered time-horizon, $b > 0$ is the finite bandwidth constraint, d_i is the limited communication times for each sensor i. Apparently, this more general problem setup includes the current paper as a special case, i.e., when $N = b = 1$ and $d_1 = d$. An explicit optimal schedule to this general problem, however, is far from clear at the present stage. It will be worthwhile and interesting to consider this problem in the future work.

V. Examples

Since the result on the terminal cost function is quite straightforward, we only provide here an example for the average cost function case. Consider $a = 1.05$, $c = 3$, $q = 2$, $r = 10$, and $p_0 = 1$. If $T = 21$ and $d = 1$, Theorem 4.1 states that the optimal transmission time is $k^\star = 11$, which is indeed verified from Fig. 2, where the left part plots $J(\theta) = J(k)$ as a function of the transmission time k. If $T = 20$ and $d = 2$, Theorem 4.3 states that the optimal transmission times are $k_1^\star = 7$ and $k_2^\star = 14$. Again this is verified from the right part of Fig. 2.

We now show how to obtain the optimal schedule analytically. The proof of Theorem 4.2 provides us a way to obtain the optimal schedule from an arbitrary initial schedule. For instance, let us start with the schedule $k_1 = 1, k_2 = 20$, i.e., $\omega_1 = 0, \omega_2 = 18, \omega_3 = 0$. Clearly this schedule is not optimal as ω_1 is not within $\omega_2 - 1$ and $\omega_2 + 1$. According to the proof of Theorem 4.2, we can shift k_1 forward by one step and construct a schedule with $\omega_1 = 1, \omega_2 = 17, \omega_3 = 0$, which has been shown to have a lower cost. Continuing to shift k_1 forward several times, we produce the schedule with $\omega_1 = 9,\ \omega_2 = 9,\ \omega_3 = 0$. This, however, is not optimal as ω_3 is not within $\omega_2 - 1$ and $\omega_2 + 1$. Thus, we can improve this schedule by shifting k_2 backward one step and obtain the schedule with $\omega_1 = 9,\ \omega_2 = 8,\ \omega_3 = 1$. Continuing to shift k_2 backward several times, we construct a schedule with $\omega_1 = 9,\ \omega_2 = 5,\ \omega_3 = 4$. Repeatedly the above process, we eventually obtain the schedule with $\omega_1 = 6,\ \omega_2 = 6,\ \omega_3 = 6$. This last schedule which corresponds to that $k_1^* = 7$ and $k_2^* = 14$ is optimal by Theorem 4.3.

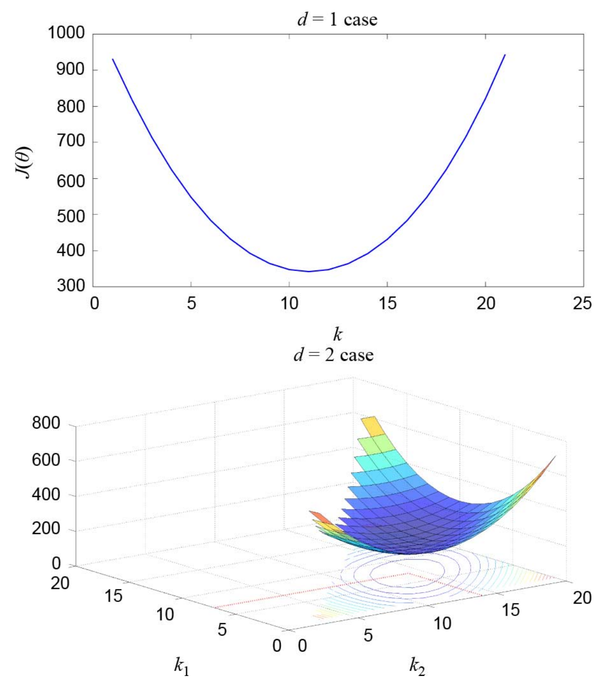

Fig. 2. The left one shows that the schedule with $k = 11$ is optimal when $T = 21, d = 1$. The right one is the $d = 2$ case. The center of the contour is the projection of the minimum point, which is (7,14).

VI. Conclusion

In this paper we study the system state estimation problem with limited measurements within a finite horizon T. We show that the times of transmitting measurement in the optimal schedule should be close to uniform distribution. In the special case, the transmitting times are exactly uniformly distributed. Future work include searching for the optimal schedule for a general T and extending current results to higher order Gauss–Markov systems as well as systems involving multiple sensors.

Appendix

Proof to Lemma 2.1: Let $\Delta(x) = a^2x+q+\frac{r}{c^2}$, then one obtains $g(x) = \frac{r}{c^2} - (\frac{r}{c^2})^2\frac{1}{\Delta(x)}$. Also define $\Gamma(x) = a^4x + a^2q + q + \frac{r}{c^2}$. Then

$$
\begin{aligned}
&hg(x) - gh(x) \\
&= a^2\frac{r}{c^2} + q - \left(\frac{r}{c^2}\right)^2\frac{a^2}{\Delta(x)} - \left(\frac{r}{c^2} - \left(\frac{r}{c^2}\right)^2\frac{1}{\Gamma(x)}\right) \\
&= \frac{r}{c^2}\,\frac{\left(a^2 - 1 + q(\frac{r}{c^2})^{-1}\right)\Delta(x)\Gamma(x) + \frac{r}{c^2}\Delta(x) - a^2\frac{r}{c^2}\Gamma(x)}{\Delta(x)\Gamma(x)}.
\end{aligned}
$$

Since $|a| \geq 1$, $a^2 - 1 \geq 0$, straightforward computation shows that $hg(x) - gh(x) > 0$. ■

Proof to Theorem 4.1: First consider $T = 2m+1$. Comparing two schedules $\theta_1 = \{k_1\}$ and $\theta_2 = \{k_2\}$ with $k_1 = k_2 - 1$, we have the following two cases.

Case 1 $\quad 1 \leq k_1 \leq m$.

$$
\begin{aligned}
&J(\theta_1) - J(\theta_2) \\
&= \left(\sum_{i=1}^{k_1-1} h^i(p_0) + \sum_{i=0}^{T-k_1} h^i g h^{k_1-1}(p_0)\right) \\
&\quad - \left(\sum_{i=1}^{k_2-1} h^i(p_0) + \sum_{i=0}^{T-k_2} h^i g h^{k_2-1}(p_0)\right) \\
&= \sum_{i=0}^{T-k_1} h^i g h^{k_1-1}(p_0) - h^{k_2-1}(p_0) - \sum_{i=0}^{T-k_2} h^i g h^{k_2-1}(p_0) \\
&= h^{T-k_1} g h^{k_1-1}(p_0) - h^{k_2-1}(p_0) \\
&\quad + \sum_{i=0}^{T-k_2}\left(h^i g h^{k_1-1}(p_0) - h^i g h^{k_2-1}(p_0)\right).
\end{aligned}
$$

Since $T - k_1 \geq m + 1 > k_1$,

$$
\begin{aligned}
&h^{T-k_1} g h^{k_1-1}(p_0) - h^{k_2-1}(p_0) \\
&\quad = a^{2(T-k_1)} g h^{k_1-1}(p_0) - a^{2(k_2-1)}p_0 + q\left(\sum_{i=k_2-1}^{T-k_1-1} a^{2i}\right) \\
&\quad \geq \frac{1}{2}a^{2(T-k_1)}\frac{r}{c^2} - a^{2k_1}\frac{r}{c^2} + q\left(\sum_{i=k_1}^{T-k_1-1} a^{2i}\right).
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
&\sum_{i=0}^{T-k_2}\left(h^i g h^{k_1-1}(p_0) - h^i g h^{k_2-1}(p_0)\right) \\
&\quad = \sum_{i=0}^{T-k_2} a^{2i}\frac{a^{2k_1}\left(p_0 - h(p_0)\right)}{\left(\frac{c^2}{r}h^{k_1}(p_0)+1\right)\left(\frac{c^2}{r}h^{k_2}(p_0)+1\right)} \\
&\quad > -\frac{a^{2k_1}\cdot a^2 q\cdot\left(\sum_{i=0}^{T-k_2} a^{2i}\right)}{\left(\frac{c^2}{r}h^{k_1}(p_0)+1\right)\left(\frac{c^2}{r}h^{k_2}(p_0)+1\right)}
\end{aligned}
$$

where the inequality holds as

$$
\begin{aligned}
p_0 - h(p_0) &= -\left(a^2p_0 + q - p_0\right) > -\left((a^2-1)\frac{r}{c^2} + q\right) \\
&\geq -\left((a^2-1)q + q\right) = -a^2q.
\end{aligned}
$$

When $k_1 \leq m - 1$,

$$
\frac{c^2}{r}h^{k_1}(p_0) \geq \frac{c^2}{r}\left(a^{2k_1}\frac{r}{2c^2} + q\sum_{i=0}^{k_1-1}a^{2i}\right) \geq \frac{1}{2}a^{2k_1} + \sum_{i=0}^{k_1-1}a^{2i},
$$

hence

$$
\begin{aligned}
J(\theta_1) - J(\theta_2) &> \frac{1}{2}a^{2(T-k_1)}\frac{r}{c^2} - a^{2k_1}\frac{r}{c^2} + q\left(\sum_{i=k_1}^{T-k_1-1}a^{2i}\right) \\
&\quad - \frac{a^{2(k_1+1)}q\left(\sum_{i=0}^{T-k_2}a^{2i}\right)}{\beta(k_1)\beta(k_2)} \\
&\geq q\left(\sum_{i=k_1+1}^{T-k_1-1}a^{2i}\right) - \frac{a^{2(k_1+1)}q\left(\sum_{i=0}^{T-k_2}a^{2i}\right)}{\beta(k_1)\beta(k_2)} \\
&\geq a^{2(k_1+1)}q\left(\sum_{i=0}^{T-2k_1-2}a^{2i} - \frac{\sum_{i=0}^{T-k_2}a^{2i}}{\beta(k_1)\beta(k_2)}\right),
\end{aligned}
$$

where

$$
\beta(k) = \frac{1}{2}a^{2k} + \sum_{i=0}^{k-1}a^{2i} + 1. \tag{9}
$$

After some manipulation, it can be proved that $J(\theta_1)-J(\theta_2) \leq 0$. When $k_1 = m$, since

$$\begin{aligned}\frac{c^2}{r}h^{k_1}(p_0) &\geq \frac{c^2}{r}\left(a^{2k_1}\frac{r}{2c^2} + q\sum_{i=0}^{k_1-1}a^{2i}\right)\\ &= \frac{1}{2}a^{2k_1} + \frac{c^2}{r}q\sum_{i=0}^{k_1-1}a^{2i},\end{aligned}$$

we have

$$\begin{aligned}J(\theta_1)-J(\theta_2) &> \frac{1}{2}a^{2(T-k_1)}\frac{r}{c^2} - a^{2k_1}\frac{r}{c^2} + q\left(\sum_{i=k_1}^{T-k_1-1}a^{2i}\right)\\ &\quad - \frac{a^{2(k_1+1)}q\left(\sum_{i=0}^{T-k_2}a^{2i}\right)}{\gamma(k_1)\gamma(k_2)}\\ &\geq \frac{1}{2}a^{2(m+1)}\frac{r}{c^2} - \frac{a^{2(m+1)}q\left(\sum_{i=0}^{m}a^{2i}\right)}{\gamma(m)\gamma(m+1)}\\ &> a^{2(m+1)}\cdot\frac{\frac{r}{2c^2}\cdot 2\left(\frac{c^2}{r}q\sum_{i=0}^{m}a^{2i}\right) - q\left(\sum_{i=0}^{m}a^{2i}\right)}{\gamma(m)\gamma(m+1)}\\ &\geq 0,\end{aligned}$$

where

$$\gamma(k) = \frac{1}{2}a^{2k} + \frac{c^2}{r}q\sum_{i=0}^{k-1}a^{2i} + 1.$$

Since $J(\theta_1 = \{k_1\}) > J(\theta_2 = \{k_2 = k_1+1\})$ for $1 \leq k_1 \leq m$, we have the following direct result[1]

$$\begin{aligned}J(\theta = \{k=1\}) &> J(\theta = \{k=2\}) > \cdots\\ &> J(\theta = \{k = m+1\}),\end{aligned}$$

i.e., when $k \leq m+1$, $J(\theta)$ decreases as k increases.

Case 2 $\quad m+1 \leq k_1 \leq T-1$.

$$\begin{aligned}J(\theta_2)-J(\theta_1) &= \left(\sum_{i=1}^{k_2-1}h^i(p_0) + \sum_{i=0}^{T-k_2}h^igh^{k_2-1}(p_0)\right)\\ &\quad - \left(\sum_{i=1}^{k_1-1}h^i(p_0) + \sum_{i=0}^{T-k_1}h^igh^{k_1-1}(p_0)\right)\\ &= h^{k_2-1}(p_0) + \sum_{i=0}^{T-k_2}h^igh^{k_2-1}(p_0)\\ &\quad - \sum_{i=0}^{T-k_1}h^igh^{k_1-1}(p_0)\\ &= h^{k_2-1}(p_0) - h^{T-k_1}gh^{k_1-1}(p_0)\\ &\quad + \sum_{i=0}^{T-k_2}\left(h^igh^{k_2-1}(p_0) - h^igh^{k_1-1}(p_0)\right).\end{aligned}$$

Since $k_2 - 1 \geq m+1 \geq T-k_1+1$,

$$\begin{aligned}h^{k_2-1}(p_0) - h^{T-k_1}gh^{k_1-1}(p_0) &\geq h^{k_2-1}(p_0) - h^{T-k_1}(q)\\ &= h^{k_2-1}(p_0) - h^{T-k_1+1}(0)\\ &\geq 0.\end{aligned}$$

[1] $J(\theta = \{k = i\})$ stands for the cost $J(\theta)$ where $\theta = \{i\}$, i.e., under θ the sensor communicates its measurement data with the remote estimator at time $k = i$. We abuse the notation a little bit here as it is more informative.

As $h(x)$ and $g(x)$ are monotonically increasing functions, $gh^{k_2-1}(p_0) > gh^{k_1-1}(p_0)$. Hence,

$$\sum_{i=0}^{T-k_2}\left(h^igh^{k_2-1}(p_0) - h^igh^{k_1-1}(p_0)\right) > 0.$$

Therefore, $J(\theta_2) - J(\theta_1) > 0$. Similar argument as in case 1 shows that when $k \geq m+1$, $J(\theta)$ increases as k increases. Hence, we conclude that $k^\star = m+1$.

Now let us consider $T = 2m$. Similar to the discussion in the $T = 2m+1$ case, for two schedules $\theta_1 = \{k_1\}$ and $\theta_2 = \{k_2\}$ with $k_1 = k_2 - 1$, when $1 \leq k_1 \leq m-1$, we can prove that $J(\theta_1) - J(\theta_2) > 0$, and when $m+1 \leq k_1 \leq T-1$, we can prove that $J(\theta_2) - J(\theta_1) > 0$. However, when $k_1 = m, k_2 = m+1$, we have

$$\begin{aligned}&J(\theta_2) - J(\theta_1)\\ &= \sum_{i=0}^{T-k_2}h^igh^{k_2-1}(p_0) + h^{k_2-1}(p_0) - \sum_{i=0}^{T-k_1}h^igh^{k_1-1}(p_0)\\ &= \sum_{i=0}^{m-1}h^igh^{m}(p_0) + h^{m}(p_0) - \sum_{i=0}^{m}h^igh^{m-1}(p_0)\\ &= h^m(p_0) - h^mgh^{m-1}(p_0)\\ &\quad + \sum_{i=0}^{m-1}\left(h^igh^m(p_0) - h^igh^{m-1}(p_0)\right).\\ &= \frac{a^{2m}\frac{c^2}{r}\left[\left(\frac{c^2}{r}p_0 - 1\right)h^m(p_0)h^{m+1}(p_0) + h^m(p_0) + h^{m+1}(p_0)\right]}{\left(\frac{c^2}{r}h^m(p_0)+1\right)\left(\frac{c^2}{r}h^{m+1}(p_0)+1\right)}.\end{aligned}$$

Notice that $p_0 \in [\frac{r}{2c^2}, \frac{r}{c^2})$ which implies that $\frac{c^2}{r}p_0 - 1 \in [-\frac{1}{2}, 0)$. Clearly when $\frac{c^2}{r}p_0 - 1$ is sufficiently close to 0, we have $J(\theta_2) > J(\theta_1)$; and when $\frac{c^2}{r}p_0 - 1$ is close to $-\frac{1}{2}$, there exist an m and a, c, q, r such that $h^m(p_0)h^{m+1}(p_0) \gg h^{m+1}(p_0) + h^m(p_0)$, in which case we have $J(\theta_2) < J(\theta_1)$. Therefore, $J(\theta_2) - J(\theta_1)$ is not always positive or negative for all p_0. By solving $J(\theta_2) - J(\theta_1) > 0$ and $J(\theta_2) - J(\theta_1) < 0$ we have the range of p_0. One sufficient condition for $J(\theta_2) - J(\theta_1) \geq 0$ is $p_0 \geq gh^{m-1}(p_0)$, e.g., $p_0 \geq \bar{p}$, where $\bar{p} > 0$ satisfies $\bar{p} = gh^{m-1}(\bar{p})$. When $J(\theta_2) - J(\theta_1) \geq 0$, the optimal $k^\star = m$, and when $J(\theta_2) - J(\theta_1) \leq 0$, the optimal $k^\star = m+1$. ■

Proof To Theorem 4.2: For convenience, we denote $z_i = p_{k_i}$ and $z_0 = p_0$. Note that $\frac{r}{2c^2} \leq z_i < \frac{r}{c^2}$ for all i. First we present an overall description of the strategy for proving this theorem: assuming θ is an optimal schedule but there is at least one pair m, l of θ such that $|\omega_m - \omega_l| > 1$, which is equivalent to the following two cases: $\omega_m = s,\ \omega_l \leq s-2$ and $\omega_m = s,\ \omega_l \geq s+2$; we show that for each case, by shifting the communication time between m and l, a lower cost can be produced, hence producing a contradiction as to the optimality of θ.

Now we present the following three detailed steps to prove the theorem.

Step 1: We consider $\omega_m = s$ and $\omega_l \leq s-2$. Without loss of generality we assume $l < m$ and $\omega_l = s-2$. The case $l > m$ is equivalent to Step 2 and the cases when $\omega_l < s-2$ can be proved similarly.

Without loss of generality, we assume the intervals between ω_m and ω_l (if any) are identical to $s-1$ as other cases are easy to transfer to this one. Consider the following schedule θ' (see Fig. 3): $k_i' = k_i + 1,\ i = l, \ldots, m-1$ and $k_i' = k_i$

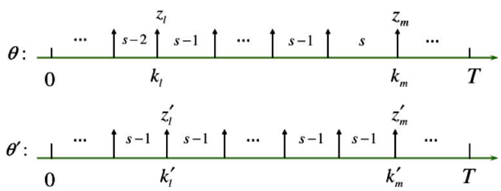

Fig. 3. For a schedule θ with $\omega_m = s, \omega_l = s-2, l < m-1$, we construct θ' by shifting $k_l, \ldots, k_{m-1}$ one step forward.

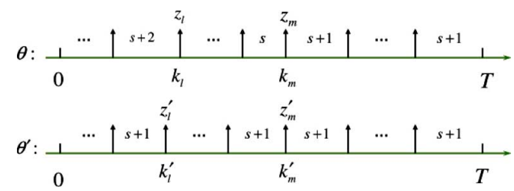

Fig. 4. For a schedule θ with $\omega_m = s, \omega_l = s+2$, we construct θ' by shifting $k_l, \ldots, k_{m-1}$ one step backward.

for all other i's. Then it is straightforward to verify that $\omega_l' = \omega_l + 1 = s-1, \omega_m' = \omega_m - 1 = s-1$, and $\omega_i' = \omega_i,\ i \neq m, l$. Hence,

$$\begin{aligned}
J(\theta) - J(\theta') &= \sum_{k=1}^{T} p_k - \sum_{k=1}^{T} p_k' \\
&= \sum_{j=l}^{d} \sum_{i=0}^{\omega_{j+1}} h^i(z_j) - \left(h^{\omega_l'}(z_{l-1}') + \sum_{j=l}^{d} \sum_{i=0}^{\omega_{j+1}'} h^i(z_j') \right) \\
&= h^{\omega_m}(z_{m-1}) - h^{\omega_l'}(z_{l-1}') \\
&\quad + \sum_{j=l}^{m-1} \sum_{i=0}^{\omega_{j+1}} \left(h^i(z_j) - h^i(z_j') \right) \\
&\quad + \sum_{j=m}^{d} \sum_{i=0}^{\omega_{j+1}} \left(h^i(z_j) - h^i(z_j') \right) \\
&\geq h^s(z_{m-1}) - h^{s-1}(z_{l-1}') \\
&\quad + \sum_{j=l}^{m-1} \sum_{i=0}^{\omega_{j+1}} \left(h^i(z_j) - h^i(z_j') \right) \\
&= a^{2s} z_{m-1} + a^{2(s-1)} q - a^{2(s-1)} z_{l-1}' \\
&\quad + \sum_{j=l}^{m-1} \sum_{i=0}^{s-1} a^{2i} (z_j - z_j').
\end{aligned}$$

Note that $z_l' > z_l$. When $l+1 \leq j \leq m-1$,

$$\begin{aligned}
z_j - z_j' &= gh^{s-1}(z_{j-1}) - gh^{s-1}(z_{j-1}') \\
&= \frac{a^{2s}(z_{j-1} - z_{j-1}')}{\left(\frac{c^2}{r} h^s(z_{j-1}) + 1\right)\left(\frac{c^2}{r} h^s(z_{j-1}') + 1\right)} \\
&= \frac{(a^{2s})^{j-l}(z_l - z_l')}{\prod_{n=l}^{j-1} \left(\frac{c^2}{r} h^s(z_n) + 1\right)\left(\frac{c^2}{r} h^s(z_n') + 1\right)} \leq 0.
\end{aligned}$$

Then $z_j - z_j' \geq -\frac{(a^{2s})^{j-l}(z_l' - z_l)}{(\beta(s))^{2(j-l)}}$, where $\beta(s)$ is given by (9). Therefore,

$$\begin{aligned}
J(\theta) - J(\theta') \geq \frac{1}{2} a^{2s} \frac{r}{c^2} + a^{2(s-1)} \left(q - \frac{r}{c^2}\right) \\
-(z_l' - z_l) \left(\sum_{i=0}^{s-1} a^{2i} \right) \sum_{j=l}^{m-1} \frac{(a^{2s})^{j-l}}{(\beta(s))^{2(j-l)}}.
\end{aligned}$$

Note that

$$\begin{aligned}
z_l' - z_l &= gh^{s-1}(z_{l-1}) - gh^{s-2}(z_{l-1}) \\
&= \frac{a^{2(s-1)} \left(h(z_{l-1}) - z_{l-1}\right)}{\left(\frac{c^2}{r} h^s(z_{l-1}) + 1\right)\left(\frac{c^2}{r} h^{s-1}(z_{l-1}) + 1\right)} \\
&< \frac{a^{2(s-1)} \left(a^2 \frac{r}{c^2} - \frac{r}{c^2} + q\right)}{\beta(s)\beta(s-1)},
\end{aligned}$$

where we use the fact that $h(z_{l-1}) - z_{l-1} = a^2 z_{l-1} + q - z_{l-1} < (a^2 - 1)\frac{r}{c^2} + q$ and

$$\sum_{j=l}^{m-1} \frac{(a^{2s})^{j-l}}{(\beta(s))^{2(j-l)}} \leq \frac{(\beta(s))^2}{(\beta(s))^2 - a^{2s}}.$$

Therefore,

$$\begin{aligned}
J(\theta) - J(\theta') &> \frac{1}{2} a^{2s} \frac{r}{c^2} + a^{2(s-1)} \left(q - \frac{r}{c^2}\right) \\
&\quad - \frac{a^{2(s-1)} \left(a^2 \frac{r}{c^2} - \frac{r}{c^2} + q\right) \sum_{i=0}^{s-1} a^{2i}}{\beta(s-1)} \\
&\quad \cdot \frac{\beta(s)}{(\beta(s))^2 - a^{2s}} \\
&\geq \frac{1}{2} a^{2s} \frac{r}{c^2} + a^{2(s-1)} \left(q - \frac{r}{c^2}\right) \\
&\quad - a^{2(s-1)} \left(a^2 \frac{r}{c^2} - \frac{r}{c^2} + q\right) \cdot a^2 \\
&\quad \cdot \frac{\beta(s)}{(\beta(s))^2 - a^{2s}} \\
&= a^{2s} \frac{r}{c^2} \left(\frac{1}{2} - a^2 \frac{\beta(s)}{(\beta(s))^2 - a^{2s}} \right) \\
&\quad + a^{2(s-1)} \left(q - \frac{r}{c^2}\right) \left(1 - a^2 \frac{\beta(s)}{(\beta(s))^2 - a^{2s}}\right),
\end{aligned}$$

where the second inequality follows from $\frac{\sum_{i=0}^{s-1} a^{2i}}{\beta(s-1)} \leq a^2$. Simple calculation reveals that the content in the first largest bracket is non-negative. Therefore, $J(\theta) - J(\theta') > 0$.

Step 2: We consider $\omega_m = s$ and $\omega_l \geq s+2$. Without loss of generality we assume $l < m$ and $\omega_l = s+2$. The case $l > m$ is equivalent to Step 1 and the cases when $\omega_l > s+2$ can be proved similarly.

We can construct a schedule θ' as follows: $k_i' = k_i - 1,\ i = l, \ldots, m-1$, and $k_i' = k_i$ for other i's. Then $\omega_l' = \omega_l - 1, \omega_m' = \omega_m + 1$ and $\omega_i' = \omega_i,\ i \neq m, l$. Using the same technique from Step 1, it is not difficult to show that $J(\theta) - J(\theta') > 0$. Thus, θ' is better than θ, which contradicts the optimality of θ.

Step 3: Write $\omega_{d+1} = s$. From the first two steps, all the other intervals should be $s+1$ or $s-1$. Furthermore, intervals with length $s-1$ and $s+1$ can not coexist. Hence, the length of the intervals of an optimal schedule should be s and $s-1$ or s and $s+1$. Since $T = (d+1)M + L$, the intervals should have length M and $M-1$. From the equalities $N_M + N_{M-1} = d+1$ and $N_M M + N_{M-1}(M-1) + d = T$, we arrive at $N_{M-1} = d - L$ and $N_M = L + 1$. ■

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On the Restricted Neyman–Pearson Approach for Composite Hypothesis-Testing in Presence of Prior Distribution Uncertainty

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***Abstract*—The restricted Neyman–Pearson (NP) approach is studied for composite hypothesis-testing problems in the presence of uncertainty in the prior probability distribution under the alternative hypothesis. A restricted NP decision rule aims to maximize the average detection probability under the constraints on the worst-case detection and false-alarm probabilities, and adjusts the constraint on the worst-case detection probability according to the amount of uncertainty in the prior probability distribution. In this study, optimal decision rules according to the restricted NP criterion are investigated. Also, an algorithm is provided to calculate the optimal restricted NP decision rule. In addition, it is shown that the average detection probability is a strictly decreasing and concave function of the constraint on the minimum detection probability. Finally, a detection example is presented to investigate the theoretical results, and extensions to more generic scenarios are provided.**

***Index Terms*—Composite hypothesis, hypothesis-testing, max-min, Neyman–Pearson (NP), restricted Bayes.**

I. INTRODUCTION

Bayesian and minimax hypothesis-testings are two common approaches for the formulation of testing [1, pp. 5–22]–[3]. In the Bayesian approach, all forms of uncertainty are represented by a prior probability distribution, and the decision is made based on posterior probabilities. On the other hand, no prior information is assumed in the minimax approach, and a minimax decision rule minimizes the maximum of risk functions defined over the parameter space [1, pp. 13–22], [4]. The Bayesian and minimax frameworks can be considered as two extreme cases of prior information. In the former, perfect (exact) prior information is available whereas no prior information exists in the latter. In practice, having perfect prior information is a very exceptional case [5]. In most cases, prior information is incomplete and only partial prior information is available [5], [6]. Since the Bayesian approach is ineffective in the absence of exact prior information, and since the minimax approach, which ignores the partial prior information, can result in poor performance due to its conservative perspective, there have been various studies that take partial prior information into account [5]–[11], which can be considered as a mixture of Bayesian and frequentist approaches [12]. The most prominent of these approaches are the empirical Bayes, Γ-minimax, restricted Bayes and mean-max approaches [5]–[7], [11], [13]. As a solution to the impossibility of complete subjective specification of the model and the prior distribution in the Bayesian approach, the robust Bayesian analysis has been proposed [12], [14, pp. 195–214]. Although the robust Bayesian analysis is considered purely in the Bayesian framework in general, it also has strong connections with the empirical Bayes, Γ-minimax and restricted Bayes approaches [12], [14, pp. 215–235].

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